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Galois coactions for algebraic and locally compact quantum groups

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Ce ne sont pas ces dons-là, pourtant, ni l'ambition même la plus ardente, servie par une volonté sans failles, qui font franchir ces "cercles invisibles et impérieux" qui enferment notre Univers. Seule l'innocence les franchit, sans le savoir ni s'en soucier, en les instants où nous nous retrouvons seul à l'écoute des choses, intensément absorbé dans un jeu d'enfant...

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Contents

Introduction

This work grew out of an attempt to better understand some of the concepts behind the thesis 'Monoidal equivalence of compact quantum groups' by A. De Rijdt. Motivated by remarks of A. Van Daele, I wanted to see if some of the ideas and constructions treated there could be put into the framework of algebraic quantum groups. This resulted in the paper [19]. Once this was achieved, it was natural to extend our results to the level of locally compact quantum groups, which became the paper [18]. The two parts of this thesis can be seen as extended versions of these papers, provided with some more motivation and introductory material.

I will now explain the main concepts involved in this thesis.

Quantum groups and Hopf algebras

The term 'quantum group' covers a broad range of many particular instances, each with their own distinct flavor. As examples, we mention Hopf algebras, quasi-Hopf algebras, quasi-triangular Hopf algebras, multiplier Hopf algebras, algebraic quantum groups, compact quantum groups, locally compact quantum groups, ... The most widely known among these would be the Hopf algebras, whose formal definition dates from the fifties. Hopf's name has been attached to these objects since cruder forms of their structure appeared first, implicitly, in his paper [47], where the cohomology groups of H-spaces (topological spaces with a multiplication map) are studied. We refer to [2] for a recent historical survey of the emergence of the concept of a Hopf algebra. We further mention the books [81] and [1], which treat the basic theory of Hopf algebras.

Geometrically, Hopf algebras are to be seen as function spaces on 'quantum affine group schemes': they are unital, not necessarily commutative algebras (over a field, or more generally, over a commutative ring), which in addition carry structures called 'comultiplication', 'counit' and 'antipode'. These respectively play the rôle of 'group multiplication', 'unit in the group' and 'taking the inverse of an element'. Although these classical analogies are very helpful for intuition, one should not expect the passage to be without surprises: for example, the operation of inversion will not be involutive for a general Hopf algebra!

In the eighties, the Leningrad school developed an important class of examples of quantum groups, which came forth naturally from their study of quantum integrable systems. This class contained for example the q-deformations of (the enveloping algebra of the Lie algebra of) semi-simple Lie groups, the q being some complex number or formal parameter which deforms the classical structure. At around the same time, S.L. Woronowicz introduced the notion of a compact quantum group ([103], [104]), which was a non-commutative topological object (C*-algebra) *containing* a dense Hopf algebra (with some further structure). This theory turned out to have much in common with the beautiful classical theory of compact groups: one is able to construct from the bare bones axiom system an analogue of the Haar measure, one can generalize the Peter-Weyl representation theory,... But there are also some new phenomena which appear. For example, because the antipode of the quantum group does not have to be involutive, it is in some cases possible to assign canonically to a representation of the quantum group a positive *non-integer number*, called quantum dimension, which still has all the expected properties of a dimension function.

In [69], M. Rosso showed how the abstract theory of compact quantum groups could be reconciled with the examples of the Leningrad school. Then S. Wang, in [102], discovered examples of compact quantum groups (called *free* compact quantum groups), which were of a different type than the q-deformations, and which turned out to have deep connections with the theory of free probability, as developed by Voiculescu. It were precisely these free quantum groups which were the subject of [26]: there it was shown that, at least for a certain class of the free quantum groups, there is still a connection with the q-deformed Lie groups: one could find a *monoidal equivalence* between a compact quantum group of this class and a particular q-deformed Lie group. We give some intuition concerning this notion of monoidal equivalence in the following paragraphs.

Monoidal equivalence

Given a compact group, one can consider its category of finite dimensional unitary representations. This is a highly structured category: each endomorphism space is a finite dimensional matrix algebra, one can multiply representations (in a functorial way) by taking a tensor product (i.e., one has a *monoidal* structure), one can 'invert' a representation by taking its contragredient, and one can let representations trade places in a tensor product representation by a canonical symmetry. A very beautiful, deep and powerful theorem of Doplicher and Roberts ([28]) has as a corollary, that if one would be given such a category with all the mentioned structure, *one can reconstruct the compact group* (see also the corresponding theorem by Deligne concerning algebraic groups and more general finite-dimensional representations, [24]).

When considering Hopf algebras or compact *quantum* groups, the representation category (of the 'underlying quantum group') still has a lot of structure: only the symmetry is missing, because of the non-commutativity of the 'function algebra'. Saying that two Hopf algebras or compact quantum groups are monoidally equivalent ([71], resp. [10]), is then precisely this notion of 'having the 'same' (C*-)category with the 'same' monoidal structure' (we will give more rigorous definitions of 'sameness' in the first chapter, but only in the non-*-setting). This provides then a very natural and strictly weaker notion of 'equality' between quantum groups. In particular, the monoidal category alone is not sufficient to reconstruct the quantum group. (In fact, the same is already true for finite groups: two non-isomorphic finite groups can have the same monoidal category (in the absence of a *-structure, see [35], in the presence of a *-structure, see [48]), but then necessarily the corresponding symmetry transformation is different.)

Galois objects

There is another, more concrete way to capture the notion of 'being monoidally equivalent', which we will now discuss. Given two monoidally equivalent Hopf algebras (or compact quantum groups), one has, by definition, a monoidal equivalence between their categories of representations, but such an equivalence need not be unique. It turns out that each equivalence itself has considerable structure: it is implemented by an algebra which carries commuting actions by both quantum groups. The algebra and the actions will satisfy some specific properties, which can be abstractly characterized. One finds then that *the datum of one of the quantum groups becomes (almost) superfluous*: one can reconstruct it, together with its action, simply given the algebra and the other quantum group (with its action). This provides one with a way of actually constructing *new* concrete quantum groups from old ones, by finding such particular algebras.

As said, the action of the quantum group on such an algebra has to be of a special type. It will be a particular instance of a *Galois action* of a quantum group on a quantum space. These Galois actions have very nice geometrical descriptions: for example, if one translates the defining conditions to the setting of locally compact groups and spaces, one ends up with the notion of a *free* and *Cartan* action[1]. A prime example of such an action is the one on a principal fiber bundle by its structure group. But even in the purely algebraic realm, these Galois actions naturally appear: if one considers a finite Galois extension of fields, then the action of the automorphism group of the extension on the big field will indeed be Galois in this sense. So this notion crops up in different places of mathematics, and in fact, many generalizations of this concept have already been considered (for an example and some discussion concerning these generalizations, see [40]).

The peculiarity of the Galois actions which provide monoidal equivalences between Hopf algebras, is that they are also *transitive* (or ergodic, depending on the context). If one would translate this condition again to the classical, geometrical setting, one would end up with something which, at first sight, appears to be quite trivial: for if a group acts free *and* transitively on a space, then this space must necessarily be set-isomorphic to the group, the action then being given by (say) right translation. The important point to make however is that *the isomorphism is not a canonical one*! For example, there is a conceptual difference between the plane, considered as an affine space, and the abelian group $\mathbb{R}^2$, which however acts on it in a free and transitive way: in the plane, there is no distinguished origin. This is why, even in the classical case of groups, spaces carrying a free and transitive

[1]For this terminology, see [64]. Briefly, 'free and Cartan' means that the group G acts continuously on the space X, in such a way that $X \times G$ is homeomorphic (via the natural map) to the equivalence relation induced on the space X, seen as a subset of $X \times X$ with the trace topology. If moreover this equivalence relation is *closed* in $X \times X$ (or, equivalently, if the orbit space X/G is Hausdorff), one calls the action 'free and proper'.

action have procured a special name for themselves, namely 'torsors'. In the quantum context, this difference becomes more than merely conceptual, since 'quantum torsors' (i.e. the objects underlying a Galois object) can have a different 'function algebra' than the quantum group itself.

The non-compact case

So far, we have only considered *compact* quantum spaces, which in algebra terms means that all algebras concerned are unital. The main purpose of this thesis is to extend the theory of (ergodic) Galois coactions to the non-compact setting. We do this both in the purely algebraic setting, generalizing 'well-behaving' Hopf algebras to algebraic quantum groups, and in the analytic setting, generalizing compact quantum groups to locally compact quantum groups. We give some information about these structures.

Algebraic quantum groups were defined and studied by A. Van Daele in [93], building upon the work done in [92]. In the latter article, a genuine generalization of Hopf algebras was introduced, the so-called 'multiplier Hopf algebras'. The main observation was that for a lot of the Hopf algebra theory, one does not really need a *unital* underlying algebra. Algebraic quantum groups are then a particular class of nicely behaving multiplier Hopf algebras, namely those which have a non-trivial left-invariant functional (an analogue of the Haar measure on a locally compact group). Their structure is quite elaborate, one of the main features being that one has a duality theory: from an algebraic quantum group, one can construct its dual, and then the dual of this new object is canonically isomorphic to the original object (Pontryagin duality).

On the other hand, locally compact quantum groups, as defined by J. Kustermans and S. Vaes in [56], are purely analytic objects, living in the world of C*-algebras ('non-commutative topology') and von Neumann algebras ('non-commutative measure theory'). They are a *proper* quantized version of locally compact groups, as the locally compact quantum groups with commutative 'function algebra' are in one-to-one correspondence with locally compact groups. The theory is considered to be more or less an end-point of a long search for the right notion of a 'locally compact quantum group'. Predecessing structures which should be mentioned, and which are still interesting in their own right, are the Kac algebras (or 'ring groups' as they were originally called, see [50]), which are locally compact quantum groups of a special type, and the multiplicative unitaries ([4], or, with extra regularity conditions, [105]), which are more general than locally compact quantum groups.

It turns out that Galois objects, either for algebraic or locally compact quantum groups, inherit a lot of structure of the acting quantum group. A big part of this thesis is devoted to proving that also the reconstruction theorem, mentioned already in the context of Hopf algebras, continues to hold in these technically more challenging situations. For these reasons, one can consider (bi-)Galois objects as a kind of 'hybrid quantum groups'.

We want to end this introduction by making a remark concerning an application of ergodic Galois actions in the analytic setting, of which we do not know if it has hitherto been considered explicitly (in the most general situation) in the purely algebraic framework, namely the introduction of *projective representations for quantum groups.*

Recall that a projective unitary representation of (say) a discrete abelian group $\mathfrak{G}$ is an embedding of the group into the algebra of unitary operators on a (separable) Hilbert space $\mathscr{H}$, which preserves the multiplication *up to* a certain scalar, which will then give one a function $\Omega : \mathfrak{G} \times \mathfrak{G} \rightarrow S^1$, where S^1 is the circle, seen as complex numbers of modulus 1. Such a function Ω is called a (S^1-valued) *2-cocycle.* We note that associated to any such 2-cocycle, there is an action of the compact dual $\hat{\mathfrak{G}}$ of $\mathfrak{G}$ on a certain cocycle-twisted convolution algebra $\mathscr{L}_\Omega(\mathfrak{G})$ of $\mathfrak{G}$, making $\mathscr{L}_\Omega(\mathfrak{G})$ into a Galois object for $\hat{\mathfrak{G}}$.

Another, more intrinsic definition of a projective representation, is that it is a representation of the group into the group of *-automorphisms of $B(\mathscr{H})$, the *-algebra of all bounded operators on $\mathscr{H}$. It turns out that with the latter definition of 'projective representation', the construction mentioned in the previous paragraph, which associates to a projective representation a certain Galois object for the dual, still works in the quantum setting. However, there will in general be no associated 2-cocycle: while this notion still makes sense, it will now only appear in special cases.

Outline of the thesis

The concrete structure of this thesis is as follows.

The *first part* concerns algebraic aspects, and we have attempted to make it completely self-contained (maybe up to some minor remarks).

The first chapter begins with a quick review of the theory of Morita equivalence for unital algebras over a field. We present three alternative approaches, namely a categorical one, a concrete, symmetric one (by means of linking algebras), and a concrete, asymmetric one (by means of 'Morita modules'), and we show how one can switch between these notions. In the next section, we then put further structure on our algebras, replacing them by *Hopf* algebras, and on our Morita equivalences, replacing them by *comonoidal* Morita equivalences. One can again give differently flavored definitions of the latter concept (using the notions of a linking weak Hopf algebra and a Galois coobject), and we prove in detail the equivalence between these. In the third section, we then introduce the dual notion of a monoidal co-Morita equivalence between Hopf algebras. Here we are rather brief, since this theory has been developed in detail in a series of papers by Schauenburg (see the third section of [76] for an overview). A fourth section discusses some particular cases and examples.

The second chapter is also an introductory one. It begins with some comments on and comparisons between the regularity conditions which can be imposed on a non-unital algebra, and proceeds to explain the notion of Morita equivalence for two different kinds of non-unital algebras. We then introduce the notion of a multiplier Hopf algebra and of an algebraic quantum group, and state (mostly without proof) the main results of [92] and [93]. We also briefly state (with proof) a result which was obtained together with A. Van Daele in [21], concerning the further structure of an algebraic quantum group possessing a well-behaving *-structure. This allows for a significant simplification of some results of [53] and [55]. We end with recalling from [97] the definition of a Galois coaction for an algebraic quantum group.

The third chapter coincides more or less with the first section of our paper [19]. We examine here the further structure of Galois coactions for which the space of coinvariants coincides with the ground field (which are then called *Galois objects*). This turns out to be as rich as the structure of an algebraic quantum group: one has a notion of an antipode (squared), of invariant integrals, of modular automorphisms for them, and of a modular element linking them. Moreover, one then has commutation relations which are similar to those of algebraic quantum groups. We also comment on some special cases, namely the situation of algebraic quantum groups of discrete or compact type, and of algebraic quantum groups with a well-behaving *-structure.

In the fourth chapter, we follow the second section of [19]. We define here the notion of a linking algebraic quantum groupoid, and show that it is (essentially) dual to the notion of a Galois object by some concrete 'Pontryagin duality' functor. In particular, we can show then that the main result of [71] holds in our setting: Galois objects are (essentially) the same as *bi*-Galois objects, i.e., we can canonically construct from a Galois object a (possibly different) algebraic quantum group, coacting on the same algebra, in such a way that it also becomes a Galois object for this new algebraic quantum group. We then consider again the situation where there is a *-structure present, and show that in this case the new algebraic quantum group also has a well-behaving *-structure. We end this chapter by considering a specific example.

The *second* part of our thesis concerns the analytic aspects of the theory of Galois coactions and objects, and mainly uses the language of von Neumann algebras. This part will undoubtedly be harder to follow for non-specialists, since it is more technical, and is based upon a vaster body of results from the literature.

In the *fifth chapter*, we recall some notions concerning von Neumann algebras and the associated non-commutative integration theory. We also comment on Morita theory for von Neumann algebras, and on Connes' result concerning the transportation of weights along a Morita equivalence. Most results are taken from the first chapters of [84]. The seventh section, concerning the basic construction of Jones for arbitrary operator valued weights, contains results which are probably known to specialists, but for which we have found no convenient reference in the literature.

In the *sixth chapter*, we introduce the notion of von Neumann and C*-algebraic quantum groups ([56] and [57]), the associated theory of coactions ([85]), and the notion of quantum subgroups. The fourth section contains some new results, and has to do with another viewpoint concerning some aspects of the theory of *integrable* coactions, as treated in [85]. This section will be important for the later chapters.

The *seventh chapter* is a reworking of part of our paper [20]. We begin with defining Galois coactions and Galois objects, and then proceed to develop the structure theory of the latter. These results are used in the third section to 'transport left invariant weights along monoidal correspondences'. This allows us to reflect a von Neumann algebraic quantum group, along a Galois object, to a new von Neumann algebraic quantum group. As in the purely algebraic case, we then further compare the different implementations of (co-)monoidal (co-)Morita equivalences (via bi-Galois objects or monoidal linking algebras), and explicitly make the connection with the theory of measured quantum groupoids ([59]). We end this chapter by considering the associated C*-algebraic structure.

The *eighth chapter* deals with the interplay between Galois coactions and quantum subgroups. First of all, we show that the property of being Galois is preserved under restriction to a quantum subgroup (a process which keeps the space which is acted upon the same). Next, we show that the same is true, in the special case of Galois objects, for the process of reduction (which also 'reduces' the space acted upon). Then, we show that one can induce arbitrary coactions along a bi-Galois object, thus creating a coaction for the reflected quantum group. We prove that under this induction process, the property of being Galois is preserved. Finally, we show that one can also induce a Galois object for some closed quantum subgroup to a Galois object for the bigger quantum group, and that the reflected quantum group along the original Galois object is then a closed quantum subgroup of the reflection of the bigger quantum group along the induced Galois object.

The ninth and tenth chapter contain some more specialized results.

In the *ninth chapter*, we consider the special case of *cleft* Galois objects, which are Galois objects constructed from a unitary 2-cocycle for the dual quantum group. In this case, the von Neumann algebra underlying the dual of the reflection along the Galois object coincides with the dual von Neumann algebra of the original quantum group. This allows us to compare the further structure of these duals in a more concrete way. We show for example that the scaling groups of these quantum groups are automatically cocycle equivalent (and in particular, induce the same one-parameter group in the outer automorphism group of the von Neumann algebra). We also give an easy criterion for a von Neumann algebraic quantum group to have *only* cleft Galois objects. In a second section, we then treat in the analytic context a result by Schauenburg ([71]), which elucidates in particular the nature of Galois objects for tensor products and Drinfel'd doubles.

The *tenth chapter* develops the notion of projective representations and corepresentations for quantum groups, which are closely related to Galois objects: just as, for ordinary groups, any projective representation comes together with an associated unitary 2-cocycle, so every projective corepresentation of a quantum group comes together with a Galois object. We generalize (both to the quantum and projective situation) a theorem due to Rieffel, which shows the equivalence between the square integrability of a unitary group representation (on a Hilbert space $\mathscr{H}$) and the integrability of its associated action on $B(\mathscr{H})$. We then give a specific example of an infinite-dimensional projective corepresentation of a compact quantum group, and show that if one reflects the compact quantum group along the Galois object associated to such a projective corepresentation, one will obtain a von Neumann algebraic quantum group which is no longer compact.

The *eleventh chapter* develops to some extent the C*-algebraic theory associated to measured quantum groupoids ([59] and [30]) with a finite-dimensional basis. It is included mainly to be able to give a unified account of the C*-algebraic structure pertaining to both linking and co-linking von Neumann algebraic quantum groupoids (as treated in the sixth section of the seventh chapter). Most of the results are obtained by adapting the corresponding proofs of the papers [54] and [105].

Concerning originality

Not all the results in this thesis are to be considered original, and not all new results use 'new techniques'. We therefore want to separate the wheat from the chaff here.

We first state what we believe to be the major two (surprising) results of this thesis: Theorem 7.3.7 (and its corollary 9.1.4), which states that a (generalized) cocycle twist of a locally compact quantum group is again a locally compact quantum group, without imposing any further conditions, and the example in section 10.3, which twists a compact quantum group into a non-compact quantum group.

The main new technical machinery developed to establish the mentioned theorem is collected in the Chapters 5 and 6, sections 5.7 and 6.4, and Chapter 7, sections 7.2 and 7.3. The material necessary to construct the mentioned example, and to establish its properties, is developed in the first part of chapter 9 and chapter 10.

The contents of the chapter 3 and 4, whilst having been important for me to be able to develop the analytic theory, bear too much resemblance to the theory developed in [92] and [93] to be considered really original. One of the more surprising results, concerning the existence of an 'antipode squared' on a Galois object, was originally thought to be a novel result, unknown in the Hopf algebraic theory, but I was later pointed by J. Bichon to the papers [43] and [44] by C. Grunspan and the paper [75] by Schauenburg, where one precisely considers such a notion (without actually calling it an antipode squared). Nevertheless, our definition of this map is made in a different way, which is easier to transport to the analytic setting.

The second and third section of the first chapter are also not to be considered (and are not intended to be) truly original: the second section owes much to the papers [65] (which however works almost entirely in the categorical setting) and [82], whilst the results in the third section are a blend of [71] and [8]. However, we hope at least to have brought some aspects of the theory in a novel way. For example, we are unaware of a *concrete* connection being made in the literature between the theory of Galois coobjects and the theory of weak Hopf algebras. Also the connection between Galois objects and weak Hopf algebras is only partially present in [8] (although the definition of Hopf-Galois system in that paper essentially coincides with our notion of a co-linking weak Hopf algebra).

Finally, the closing chapter 11 contains generalizations of the results of [54] and [105]. Most of its proofs however can more or less be copied from these papers, with minor modifications here and there.

We also want to comment on the originality of the *concepts* used. There are two notions which we think deserve attention.

First of all, we have prominently used the notion of a linking structure wherever possible. This seems not to be used much in the pure algebra setting (where one likes to work more with the equivalent notion of a Morita context), but it is a familiar concept to operator algebraists (see e.g. [67] and [39]) and groupoid theorists (see e.g. [17]). The benefit of using a linking structure is that it has a similar structure as the objects which it links, so that one obtains a more unified picture than when considering the constituents of a linking structure separately. For example, our definition of a co-linking weak Hopf algebra coincides with the (piecewise) definition of a Hopf-Galois system of [10], but while the latter definition seems rather complicated at first sight, our definition seems more natural, since it simply concerns weak Hopf algebras with a distinguished projection.

Another notion which we believe to be new and of importance, is that of a projective (co-)representation for a quantum group. Indeed: this could even be seen as the real motivation for considering Galois objects in an analytic setting, for there is a one-to-one correspondence between (outer equivalence classes of) coactions of a locally compact quantum group M on type I-factors (i.e. von Neumann algebras of the form $B(\mathscr{H})$ for some Hilbert space $\mathscr{H}$), and (isomorphism classes of) Galois objects for its dual $\widehat{M}$ (see Theorem 10.1.3).

Notations

This is a list of the notations which we will frequently use throughout the thesis.

When S is a set, we denote by ι_S the identity map on the set S. More generally, when $\mathcal{C}$ is a category, we denote the identity morphism of an object S by ι_S. The symbol $\circ$ denotes the composition of maps (or morphisms), but we mostly suppress it. If f is a map (or more generally a morphism), we denote its domain by $\mathscr{D}(f)$.

Throughout the first part of this thesis, k will denote an arbitrary field, except at those places where it is specifically stated that we take $k = \mathbb{C}$. By $M_n(k)$, we denote the algebra of n-by-n-matrices over k. If V, W are vector spaces over k, we denote by $V \odot W$ the tensor product of V and W over k, and we write the elements of $V \odot W$ as $\sum_i v_i \otimes w_i$. We also write the tensor product of linear maps x and y as $x \otimes y$. If A is an algebra, V a right A-module and W a left A-module, we denote by $V \underset{A}{\odot} W$ the balanced tensor product, and by $v \underset{A}{\otimes} w$ a simple tensor inside. When A, B are C*-algebras, we denote by $A \underset{\min}{\otimes} B$ their minimal tensor product. When M, N are von Neumann algebras, we denote by $M \otimes N$ their spatial tensor product. We also denote the Hilbert space tensor product of two Hilbert spaces $\mathscr{H}$ and $\mathscr{G}$ as $\mathscr{H} \otimes \mathscr{G}$. Then if ξ is a vector in $\mathscr{H}$, we denote

$$l_\xi : \mathscr{K} \to \mathscr{H} \otimes \mathscr{K} : \eta \to \xi \otimes \eta,$$

and if η is a vector in $\mathscr{K}$, we denote

$$r_\eta : \mathscr{H} \to \mathscr{H} \otimes \mathscr{K} : \xi \to \xi \otimes \eta.$$

When $\mathscr{K} = \mathbb{C}$, we identify $\mathscr{H} \otimes \mathbb{C}$ and $\mathbb{C} \otimes \mathscr{H}$ with $\mathscr{H}$, and we then denote $l_\xi^* = r_\xi^*$ as ω_ξ.

When V, W, Z are vector spaces, and

$$\cdot : V \times W \to Z : (v, w) \to v \cdot w$$

a bilinear map, we denote for $A \subseteq V$ and $B \subseteq W$:

$$A \cdot B := \{\sum_i v_i \cdot w_i \mid v_i \in A, w_i \in B\} \subseteq Z.$$

When $v \in V$, we then also write $v \cdot B := \{v\} \cdot B$.

By $\Sigma_{V,W}$, or simply Σ when V and W are clear from the context, we denote the flip map between two vector spaces V and W:

$$\Sigma_{V,W} : V \odot W \to W \odot V : \sum_i v_i \otimes w_i \to \sum_i w_i \otimes v_i.$$

We will also frequently use the leg numbering notation: if V_i are vector spaces and

$$u : V_1 \odot V_2 \to V_3 \odot V_4$$

is a linear map, we denote for example by u_{12} the linear map

$$u \otimes 1 : V_1 \odot V_2 \odot V_5 \to V_3 \odot V_4 \odot V_5,$$

and by u_{13} the linear map

$$\Sigma_{23} u_{12} \Sigma_{23} : V_1 \odot V_5 \odot V_2 \to V_3 \odot V_5 \odot V_4.$$

If u is already indexed, say $u = u_1$, then we write $u_{1,13}$ for u_{13}. We also use the same notations when working with Hilbert spaces instead of just vector spaces.

The scalar product of a Hilbert space will be anti-linear in the second argument. If $\mathscr{H},\mathscr{K}$ are Hilbert spaces, we denote by $B(\mathscr{H},\mathscr{K})$ the Banach space of all bounded operators between $\mathscr{H}$ and $\mathscr{K}$, by $B(\mathscr{H})$ the algebra of all bounded operators on $\mathscr{H}$, and by $B_0(\mathscr{H})$ the algebra of all compact operators. If $\xi,\eta \in \mathscr{H}$, we write

$$\omega_{\xi,\eta} : B(\mathscr{H}) \to \mathbb{C} : x \to \langle x\xi, \eta\rangle.$$

If u is a unitary on $\mathscr{H}$, we will denote

$$\mathrm{Ad}(u) : B(\mathscr{H}) \to B(\mathscr{H}) : x \to uxu^*.$$

If ω is a functional on $B(\mathscr{H})$ (or any other *-algebra), we denote $\overline{\omega}(x) = \overline{\omega(x^*)}$.

Unbounded positive operators on a Hilbert space $\mathscr{H}$ are always assumed to be self-adjoint (in particular, closed and densily defined). When $x \in B(\mathscr{H})$ and A a positive operator, we call x a left (resp. right) multiplier of A if xA (resp. Ax) is bounded. We then write xA (resp. Ax) also for the closure of this map.

Most of the time, we will only work with structures imposed on a vector space, and *we will then denote the whole structure by just the symbol for this underlying vector space.* This will not lead to any confusion, since we will always use standardized symbols for the extra structure, indexed by the underlying vector space. When we put two structures on the same vector space, we will then use another symbol to denote the same vector space.

Algebra

Chapter 1

Morita theory for Hopf algebras

This chapter is meant as an easily accessible introduction to the notions of 'comonoidal Morita equivalence' and 'monoidal co-Morita equivalence' in the setting of Hopf algebras. The monoidal co-Morita theory is well-developed in the literature (see [76], section 3 for a nice overview), whereas the comonoidal Morita theory seems not to have been examined in full detail (although the results are not very surprising, given that, at least formally, they are dual to the ones of the monoidal co-Morita theory. See also [65], [82] and section 4 of [78] for some discussion in quite different contexts). Therefore, we spend some time on developing the latter theory (which is quite convenient for introductory purposes, since it builds upon the better known notion of Morita equivalence between unital algebras), while for the former theory, we mostly just state the results, and refer to the literature for proofs.

1.1 Morita theory for algebras

1.1.1 Unital algebras

Definition 1.1.1. *We call a couple* (A, M_A) *an* associative k-algebra *if* A *is a non-zero vector space over* k *equipped with a* k*-linear map*

$$M_A : A \odot A \to A$$

which satisfies the following associativity relation:

$$M_A(M_A \otimes \iota_A) = M_A(\iota_A \otimes M_A)$$

as maps $A \odot A \odot A \to A$.

We say that A *has a* unit *or* is unital *when there exists an element* $1_A \in A$ *such that*

$$1_A \cdot a = a = a \cdot 1_A \qquad \text{for all } a \in A.$$

Since associative k-algebras are the only types of algebras we will work with, we will use the abbreviated form 'algebra' for them. *Also, as mentioned at the end of the introduction, we will from now on denote algebras by just the symbol for the underlying vector space.*

Multiplication in an algebra A is as usual just denoted by a dot, or no symbol at all:

$$aa' = a \cdot a' := M_A(a \otimes a') \qquad \text{for } a, a' \in A.$$

Note that a unit in a unital algebra is unique, and hence we may talk about *the* unit. When A is a unital algebra, we denote then by η_A the map

$$k \to A : c \to c1_A,$$

which we will call the *unit map*. Note then that η_A satisfies the identities

$$M_A(\eta_A \otimes \iota_A) = \iota_A = M_A(\iota_A \otimes \eta_A),$$

where we have canonically identified $k \odot A$ and $A \odot k$ with A.

Definition 1.1.2. *Let* A *be an algebra. The* opposite algebra *is the algebra* $A^{op} = (A, M_A \circ \Sigma_{A,A})$. *We will write an element* a *of* A *as* a^{op} *when we consider it as an element of* A^{op}.

Definition 1.1.3. *Let* A *and* B *be algebras. The* tensor product algebra $A \odot B$ *is the algebra* $(A \odot B, (M_A \otimes M_B) \circ (\iota_A \otimes \Sigma_{B,A} \otimes \iota_B))$.

It is easily checked that the tensor product of two algebras will be unital iff both algebras are unital.

Definition 1.1.4. *Let* A *and* B *be two algebras. A* homomorphism *between* A *and* B *is a* k*-linear map* $f : A \to B$ *such that* $f(aa') = f(a)f(a')$ *for all* $a, a' \in A$. *We also call a homomorphism a* multiplicative (linear) map.

When A *and* B *are unital algebras, we call a homomorphism* $f : A \to B$ unital *if* $f(1_A) = 1_B$.

When f is a bijective homomorphism between two algebras, we call it an isomorphism*, and, when $A = B$, an* automorphism*. We call an automorphism f of a unital algebra A* inner *if there exists an invertible element $u \in A$ such that $f(x) = uxu^{-1}$ for all $x \in A$.*

We want to stress that when talking about homomorphisms between unital algebras, we mean homomorphisms of the underlying algebras. When we want them to preserve the unit, we explicitly call them *unital homomorphisms.*

When A and B are algebras, we mean by an anti-multiplicative map (or anti-homomorphism) from A to B the composition of a homomorphism from A to B^{op} with the canonical vector space isomorphism $B^{\mathrm{op}} \to B$.

1.1.2 Morita equivalence

Definition 1.1.5. *Let A be an algebra. A couple (V, m_V) consisting of a k-vector space V and a k-linear map $m_V : A \odot V \to V$ is called a* left A-module *if the equality*

$$m_V(M_A \otimes \iota_V) = m_V(\iota_A \otimes m_V)$$

between the two stated maps $A \odot A \odot V \to V$ holds.

We will again use $\cdot$, or no symbol at all, to denote the action of A on V, i.e.

$$av = a \cdot v := m_V(a \otimes v).$$

Definition 1.1.6. *Let A be an algebra, and V a left A-module.*

- *We call V* unital *if $A \cdot V = V$.*
- *We call V* faithful *if $a \cdot v = 0$ for all $v \in V$ implies $a = 0$.*

For a unital algebra A, a left module V will be unital iff $1_A \cdot v = v$ for all $v \in V$. Another way of expressing this is

$$m_V(\eta_A \otimes \iota_V) = \iota_V.$$

Associated with a unital algebra A, there is a k-abelian category[1] A-Mod. The objects of this category consist of unital left A-modules, while the morphisms $\mathrm{Mor}(V, W)$ between two objects V and W are the k-linear maps

$$x : V \to W$$

which satisfy

$$x \circ m_V = m_W \circ (\iota_A \otimes x).$$

We also call these morphisms the *intertwiners* between the two left A-modules V and W, and will denote $\mathrm{Mor}(V, W)$ as $\mathrm{Hom}_A(V, W)$.

Closely related to the notion of module is that of a *representation*. If A is a (unital) algebra, a *(unital) left representation* of A consists of a couple (V, π), where V is a k-vector space and π is a (unital) homomorphism $A \to \mathrm{End}_k(V)$. *Mostly, we will just write π for a left representation, and we write V_π for the associated vector space.* There is a one-to-one correspondence between left A-modules and left representations of A in the following way: to the left module V, we associate the left representation π_V such that $\pi_V(a)v = a \cdot v$, while to a left representation π, we associate the left A-module V_π for which m_{V_π} is the unique extension to $A \odot V$ of the k-bilinear map $A \times V \to V : (a, v) \to \pi(a)v$. This correspondence clearly preserves unitality. In the following, we will make no distinction between left modules and left representations. Note in particular that an intertwiner between two left representations π_1 and π_2 will then be a map $x : V_{\pi_1} \to V_{\pi_2}$ satisfying $\pi_2(a)x = x\pi_1(a)$ for all $a \in A$.

It is clear what the corresponding *right* notions are, and that right modules/representations of an algebra A correspond precisely to the left modules/representations of the opposite algebra A^{op}. We will denote right modules canonically by (V, n_V), and right representations by the symbol θ. We then also write

$$va = v \cdot a := \theta(a)v.$$

When A is a unital algebra, we denote the category of unital right A-modules by Mod-A.

[1]Since category theory only plays a marginal rôle in this thesis, we have decided not to include the definitions of those terms which are not essential to understand what follows, and refer to [61] for more information.

Further, if A, D are *two* (unital) algebras, then a (unital) D-A-*bimodule* consists of a vector space V which is at the same time a (unital) left D-module and a (unital) right A-module, in such a way that the two module structures commute: for al $v \in V$, $a \in A$ and $d \in D$, we have

$$d \cdot (v \cdot a) = (d \cdot v) \cdot a.$$

Note that an algebra A is itself an A-A-bimodule in a natural way.

The following is the categorical definition of a Morita equivalence of algebras:

Definition 1.1.7. *Two unital algebras A and D are called* Morita equivalent *if there exists a k-additive equivalence between Mod-A and Mod-D. The equivalence itself is called a* Morita equivalence *between A and D.*

Remark: It will follow from Proposition 1.1.12 that to any Morita equivalence, there corresponds a k-additive equivalence between A-Mod and D-Mod, so there is no left/right asymmetry.

We will now find other, more concrete ways of capturing the notion of a Morita equivalence.

We begin with the notion of a linking algebra.

Definition 1.1.8. *A unital* linking algebra *is a couple (E, e) consisting of a unital algebra E together with an idempotent $e \in E$, such that e and $1_E - e$ are* full*: $EeE = E$ and $E(1_E - e)E = E$.*

When E is a unital algebra, and e an idempotent in E, then as a vector space, E is the direct sum of vector spaces E_{ij}, where $E_{ij} = e_i E e_j$ with $e_2 = e$ and $e_1 = 1_E - e$. We mostly write this direct sum as a 2-by-2 matrix:

$$E = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix},$$

since this intuitively captures the different multiplication rules between the E_{ij}. Note that, by restricting the multiplication of E, the E_{ii} become unital algebras with unit e_i, while all E_{ij} are unital E_{ii}-E_{jj}-bimodules. The conditions for a couple (E, e), consisting of a unital algebra with an idempotent, to be a unital linking algebra can then be written as $E_{ij} \cdot E_{jk} = E_{ik}$ for all $i, j, k \in \{1, 2\}$.

Definition 1.1.9. *Let A and D be unital algebras. We call a quadruple (E, e, Φ_A, Φ_D) a* linking algebra between A and D *if (E, e) is a unital linking algebra and $A \underset{\Phi_A}{\cong} E_{22}$ and $D \underset{\Phi_D}{\cong} E_{11}$ are algebra isomorphisms.*

When we want to talk about a linking algebra between two algebras, without wanting to specify the algebras, we will talk simply of a linking algebra *between*.

One should be careful with the notion of isomorphism between linking algebras: two non-isomorphic linking algebras between can be isomorphic as unital linking algebras. Although the notions of isomorphism for the two concepts should be clear, we state them here explicitly.

Definition 1.1.10. *Let (E_1, e) and (E_2, e') be two unital linking algebras. We call them* isomorphic *if there is an algebra isomorphism $\Phi : E_1 \to E_2$ such that $\Phi(e) = e'$. If A and D are unital algebras, and $(E_1, e, \Phi_{1,A}, \Phi_{1,D})$ and $(E_2, e', \Phi_{2,A}, \Phi_{2,D})$ are linking algebras between A and D, we call them* isomorphic *if there is an isomorphism $\Phi : (E_1, e) \to (E_2, e')$ of unital linking algebras,* such that $\Phi \circ \Phi_{1,A} = \Phi_{2,A}$ *and* $\Phi \circ \Phi_{1,D} = \Phi_{2,D}$.

Most of the time, we will identify two algebras A and D with their parts inside their linking algebra (E, e) between, and suppress the symbols for the identifications.

When (E, e) is a unital linking algebra, then of course it is a linking algebra between the unital algebras E_{11} and E_{22}, by identity isomorphisms. Therefore, whenever (E, e) is a linking algebra, we will also write $E_{11} = D$ and $E_{22} = A$, whenever this is nicer to use. We then also write B for E_{12} and C for E_{21}.

We now move on to the second notion which will capture the notion of Morita equivalence in a more concrete way. In the following, if B is a right A-module for some unital algebra A, we call the module *generating* if there exist $\alpha_i \in \mathrm{Hom}_A(B_A, A_A)$ and $x_i \in B$ such that $\sum_i \alpha_i(x_i) = 1_A$. Note that a generating module is automatically faithful.

Definition 1.1.11. *Let A be a unital algebra. A right* Morita A-module *B is a non-zero unital right A-module which is projective, finitely generated and generating. If D is another algebra, we call (B, π) a* D-A-equivalence bimodule *(or an equivalence bimodule between* A *and* D*), if* B *is a right Morita* A*-module and* $\pi : D \to End_A(B_A)$ *is an isomorphism of algebras.*

Even more concretely, one can say that B is a Morita A-module iff there exist positive integers m and n, such that A is isomorphic as a right A-module to a submodule of B^m (which corresponds to the generating property), and B is isomorphic to a submodule of A^n (which corresponds to the projectivity and finite generation).

Again, there is a distinction to be made between isomorphisms of Morita modules and isomorphisms of equivalence bimodules: the isomorphism classes of equivalence bimodules can be put into a (non-canonical) 1-1-correspondence with couples consisting of an isomorphism class of a right Morita module, *together with* an element of $\mathrm{Out}(D)$, which is the group of automorphisms of D, divided out by the normal subgroup of inner automorphisms.

We also have analogous concepts in the left setting, and it is clear from the next proposition that an equivalence bimodule is really a symmetric concept: if B is a D-A-equivalence bimodule, then it is in particular a left Morita D-module.

Proposition 1.1.12. *Let* A *and* D *be unital algebras. There is a one-to-one correspondence between isomorphism classes of*

1. *Morita equivalences between* A *and* D*,*
2. *linking algebras between* A *and* D*,*
3. *equivalence bimodules between* A *and* D*.*

In particular, A *and* D *are Morita equivalent iff there exists a linking algebra between them, iff there exists an equivalence bimodule between them.*

Proof. The one-to-one-correspondence between the objects in the first and third item are well-known, while the one-to-one-correspondence between the objects of the second and third item is easy to establish directly. We therefore only present the main steps in the proof, without going in too much detail.

In the first part of the proof, we show that there are natural maps (1) $\to$ (2) $\to$ (3) $\to$ (1). In the second part, we show that these maps, on equivalence classes, are all bijections.

Let $(F, G, \eta, \varepsilon)$ be a Morita equivalence between A and D, presented as a pair of adjoint functors equipped with unit and counit natural isomorphisms η, resp. ε, where F goes from Mod-A to Mod-D. One can construct from it a D-A-equivalence bimodule as follows.

First, denote $B = G(D_D)$, which is by definition a right A-module. It is easy to see that we have a canonical isomorphism $D \to \mathrm{End}_D(D_D)$ of algebras, where an element $d \in D$ gets sent to the linear map l_d which is left multiplication with d. As G is an equivalence, we then also get a natural isomorphism $\pi : D \to \mathrm{End}_A(B_A)$.

We have to prove some properties of B_A as a right A-module. As D_D is a free right D-module, it is projective, and hence also B_A is projective. Secondly, D_D is a compact object, which means the following: whenever we have an index set I, a collection M_i of objects of Mod-D parametrized by I, and a morphism $f : D_D \to \bigoplus_{i \in I} M_i$, we can always find a finite set $I_0 \subseteq I$ such that f factorizes as $D_D \to \bigoplus_{i \in I_0} M_i \to \bigoplus_{i \in I} M_i$. Since an equivalence preserves this property, B_A is compact, and together with projectivity this implies that B_A is finitely generated. Finally, D_D is also a generating object: whenever M, N are two objects of Mod-D, and f a non-zero morphism between them, we can find a morphism $g : D_D \to M$ such that $f \circ g \neq 0$. Since this property is preserved by an equivalence, B_A will be a generating object, and from this one can deduce that B_A is a generating module.

(An alternative and rather distinct way to construct the equivalence bimodule is as follows: let U_A and U_D be the forgetful functors from Mod-A, resp. Mod-D to the abelian category k-Mod of vector spaces over k. Denote $\mathscr{B} := \mathrm{Hom}(U_D, U_A \circ G)$. It is easy to show that D can be identified with $\mathscr{D} := \mathrm{End}(U_D)$, sending $d \in D$ to the natural transformation n_d which satisfies $(n_d)_\pi = \pi(d)$. Similarly, A can be identified with $\mathrm{End}(U_A)$, and hence with $\mathrm{End}(U_A \circ G)$, since G is an equivalence. By composition of natural transformations, $\mathscr{B}$ is a $\mathscr{D}$-$\mathscr{A}$-bimodule, and hence also a D-A-bimodule. One then shows that it is an equivalence bimodule. It will be isomorphic to the previously constructed equivalence bimodule by sending $n \in \mathscr{B}$ to $b_n := (n_{D_D})(1_D)$. The proof of this last fact would be pretty similar to (and follows easily from) a later argument, which shows that G is in fact equivalent to the functor $- \underset{D}{\otimes} B$.)

We now want to create a linking algebra between A and D, directly from a D-A-equivalence bimodule B. Put $E = \mathrm{End}_A((B \oplus A)_A)$. Let $e \in E$ be the projection map onto the A_A-summand. Then it is clear that we can canonically identify $(1_E - e)E(1_E - e)$ with D. We can also identify A with eEe as an algebra, sending a to $0 \oplus l_a$, and we can identify B with $(1_E - e)Ee$ by sending b to the linear map which acts as the zero map on the B-summand, and acts as the linear map

$$A_A \to B_A : a \to b \cdot a$$

on the A-summand. Denoting $C = \mathrm{Hom}_A(B_A, A_A)$, we can also identify $eE(1_E - e)$ with C, and then write E in the form $\begin{pmatrix} D & B \\ C & A \end{pmatrix}$.

We should show that (E, e) is a unital linking algebra, which boils down to proving $B \cdot C = D$ and $C \cdot B = A$. But one easily checks that the former property holds by projectivity and finite generation, and the latter by the fact that B is generating.

Now let (E, e) be a linking algebra between A and D. Then by the A-D-symmetry of E (interchanging e and $(1_E - e)$), it is enough to prove that there is a k-linear equivalence Mod-$E \to$ Mod-A. Consider the restricting k-linear functor Res : Mod-$E \to$ Mod-A which sends a right E-representation (V, θ_V) to the right A-representation $(\theta_V(e)V, (\theta_V)_{|A})$, and the inducing k-linear functor Ind : Mod-$A \to$ Mod-E which sends a right A-module V to the right E-module $(V \underset{A}{\odot} C \quad V) := (V \underset{A}{\odot} C) \oplus V$, with E-module structure

$$(v' \underset{A}{\otimes} c' \quad v) \cdot \begin{pmatrix} d & b \\ c & a \end{pmatrix} := ((v' \underset{A}{\otimes} (c' \cdot d)) + (v \underset{A}{\otimes} c) \quad (v' \cdot (c' \cdot b)) + (v \cdot a)).$$

It is easily checked that this is a well-defined right E-module structure, and that Ind extends to a k-linear functor.

We show that Res and Ind are quasi-inverses of each other. In fact, $\mathrm{Res} \circ \mathrm{Ind}$ *is* the identity functor. On the other hand, for V a right E-module, define

$$\epsilon_V : \mathrm{Ind}(\mathrm{Res}(V)) \to V : (v \underset{A}{\otimes} c \quad v') \to v \cdot c + v'.$$

Then $V \to \epsilon_V$ is easily seen to be a natural transformation. Moreover, it is a natural isomorphism, the inverse being provided by

$$\epsilon_V^{-1} : V \to \mathrm{Ind}(\mathrm{Res}(V)) : v \to (\sum_i (v \cdot b_i) \underset{A}{\otimes} c_i \quad v \cdot e),$$

where $b_i \in B$ and $c_i \in C$ are such that $\sum_i b_i \cdot c_i = 1_D$.

It is not difficult to check that all preceding constructions descend to equivalence classes. We now show that, when applied successively, these constructions give back the same object, possibly up to isomorphism.

Let $(F, G, \eta, \varepsilon)$ be a Morita equivalence between A and D. Let $(\tilde{F}, \tilde{G}, \tilde{\eta}, \tilde{\varepsilon})$ be the Morita equivalence obtained by successively applying the previous constructions. Then $\tilde{G}$ assigns to a right D-module V_D the right A-module $V \underset{D}{\odot} (B_A)$, where $B_A = G(D_D)$. For $v \in V_D$, denote by l_v the morphism

$$l_v : D_D \to V_D : d \to v \cdot d.$$

Denote by $\tilde{\phi}_V$ the linear map

$$V \odot B \to G(V_D) : \sum_i v_i \otimes b_i \to \sum_i G(l_{v_i})(b_i),$$

which is well-defined by k-linearity of G. Then by the functoriality of G and the definition of the left D-module structure on B, $\tilde{\phi}_V$ descends to a map

$$\phi_V : V \underset{D}{\odot} B \to G(V_D),$$

and it is easy to see that ϕ is then a natural transformation, since for $g \in \mathrm{Hom}_D(V_D, W_D)$, we have $g \circ l_v = l_{g(v)}$, by right D-linearity of g.

We want to show that ϕ is a natural equivalence. We first show that each ϕ_V is injective. Choose a finite set of $b_i \in B$ which generate B as a left D-module, which is possible since B_A is generating. Suppose $v_i \in V$ are such that $\sum_i G(l_{v_i})(b_i) = 0$. Take an arbitrary $c \in C = \mathrm{Hom}_A(B_A, A_A)$ and $b \in B$. Then

$$\begin{aligned}
(\sum_i G(l_{v_i})(b_i)) \cdot (c \cdot b) &= \sum_i G(l_{v_i})(b_i \cdot (c \cdot b)) \\
&= \sum_i G(l_{v_i})((b_i \cdot c) \cdot b) \\
&= \sum_i G(l_{v_i \cdot (b_i \cdot c)})(b) \\
&= 0
\end{aligned}$$

Hence $G(\sum_i l_{v_i \cdot (b_i \cdot c)}) = 0$, and by the faithfulness of G, we get $\sum_i l_{v_i \cdot (b_i \cdot c)} = 0$. So $\sum_i v_i \cdot (b_i \cdot c) = 0$ for all c. But choosing $c'_j \in C$ and $b'_j \in B$ such that

$\sum_j c'_j \cdot b'_j = 1_A$, we get that

$$\begin{aligned}\sum_i v_i \underset{D}{\odot} b_i &= \sum_{i,j} v_i \underset{D}{\odot} b_i \cdot (c'_j \cdot b'_j) \\ &= \sum_{i,j} v_i \underset{D}{\odot} (b_i \cdot c'_j) \cdot b'_j \\ &= \sum_j (\sum_i v_i \cdot b_i \cdot c'_j) \underset{D}{\odot} b'_j \\ &= 0.\end{aligned}$$

Now we want to show that each ϕ_V is surjective. Since B_A is a generator, a similar argument as before shows that $G(V)$ is spanned as a vector space by elements of the form $f(b)$, where $b \in B$ and $f \in \mathrm{Hom}_A(B_A, G(V_D))$. But since $B_A = G(D_D)$ and G is full, such an f must be of the form $G(l_v)$ for some $v \in V$. Hence ϕ_V is surjective, and ϕ is a natural equivalence.

Now let B be a D-A-equivalence bimodule. Then the associated functor $G : \text{Mod-}D \to \text{Mod-}A$ is $V \to V \underset{D}{\odot} B$. So $G(D_D) = D \underset{D}{\odot} B \cong B$ as a D-A-bimodule.

Finally, let (E, e) be a linking algebra between A and D. Then the D-A-bimodule constructed from this is $B = (1_E - e)Ee$. Let (E', e') be the linking algebra between constructed from this. Define $\Phi : E \to E' : \begin{pmatrix} d & b \\ c & a \end{pmatrix} \to \begin{pmatrix} d & b \\ f(c) & a \end{pmatrix}$, where $f(c) \in C' = (1_{E'} - e')E'e'$ is defined uniquely by the property that $f(c) \cdot b = c \cdot b$ for all $b \in B$. Then it is easy to conclude that Φ is an isomorphism of linking algebras between A and D.

□

Corollary 1.1.13. *Let A and D be two unital algebras.*

1. *The algebras A and D are Morita equivalent iff A^{op} and D^{op} are Morita equivalent, and in this case, there is a one-to-one correspondence between isomorphism classes of their respective Morita equivalences.*

2. *Let B be an equivalence bimodule between A and D. Then B is a finitely generated projective generating left D-module.*

3. *Let E be a linking algebra between A and D. Then the natural maps $B \underset{A}{\odot} C \to D$ and $C \underset{D}{\odot} B \to A$ are isomorphisms of bimodules.*

Proof. For the first item: if (E, e) is the linking algebra between A and D associated to a Morita equivalence, then $(E^{\mathrm{op}}, e^{\mathrm{op}})$ is a linking algebra between A^{op} and D^{op}, giving the desired Morita equivalence between A^{op} and D^{op}.

For the second item: since by the first item, $B^{\mathrm{op}} := e^{\mathrm{op}} E^{\mathrm{op}}(1_{E^{\mathrm{op}}} - e^{\mathrm{op}})$ is a right D^{op}-Morita module, B will be a left D-Morita module.

Finally, the third item follows immediately from the fact that in a linking algebra E between, one can find $b_i, b'_j \in B$ and $c_i, c'_j \in C$ such that $\sum_i b_i c_i = 1_D$ and $\sum_j c'_j b'_j = 1_A$.

□

Motivated by the correspondence established in Proposition 1.1.12, we introduce the following terminology:

Definition 1.1.14. *Let A be a unital algebra. Then we call A, with its canonical A-bimodule structure, the* identity equivalence bimodule*, while we call $M_2(A) := A \odot M_2(k)$, with the canonical inclusions into the diagonal corners, the* identity linking algebra between.

When A and D are two unital algebras, and B a D-A-equivalence bimodule, we call the dual $C := Hom_A(B_A, A_A)$, with its natural A-D-bimodule structure, the inverse *of B. When (E, e) is a linking algebra between A and D, we call $(E, (1_E - e))$ the* inverse *linking algebra between D and A.*

When E_{11}, E_{22} and E_{33} are three unital algebras, and E_{12} an E_{11}-E_{22}-equivalence bimodule, E_{23} an E_{22}-E_{33}-equivalence bimodule, we call $E_{13} := E_{12} \underset{E_{22}}{\odot} E_{23}$ the composite *E_{11}-E_{33}-equivalence bimodule of E_{23} and E_{12}. When (E_1, e) is a linking algebra (between E_{11} and E_{22}), and (E_2, e') a linking algebra (between E_{22} and E_{33}), we call*

$$E = \begin{pmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{pmatrix} := \begin{pmatrix} E_{11} & E_{1,12} & E_{1,12} \underset{E_{22}}{\odot} E_{2,12} \\ E_{1,21} & E_{22} & E_{2,12} \\ E_{2,21} \underset{E_{22}}{\odot} E_{1,21} & E_{2,21} & E_{33} \end{pmatrix},$$

together with its graded structure, the associated 3×3-linking algebra *(between E_{33}, E_{22} and E_{11}), and* $\begin{pmatrix} E_{11} & E_{13} \\ E_{31} & E_{33} \end{pmatrix}$ *the composite linking algebra (between E_{33} and E_{11}).*

It is clear then that we can form a *large groupoid* with unital algebras as its objects and isomorphism classes of equivalence bimodules as morphisms, the composition being the one of the previous proposition (which is easily seen to descend to isomorphism classes), and with the identity morphisms being given by the (isomorphism class of the) identity equivalence bimodules. The inverse of (the isomorphism class of) an equivalence bimodule is then given by (the isomorphism class of) the inverse equivalence bimodule (of a representative), by the third item of Corollary 1.1.13.

We end with the following trivial proposition.

Proposition 1.1.15. *Let A be a unital algebra, B a right Morita A-module, and $D = End_A(B_A)$. Then D is a unital algebra, and B_A becomes a D-A-equivalence bimodule. In particular, A and D are Morita equivalent.*

The point is that given a right Morita A-module B, one *constructs* from it, in a canonical way, a new algebra D, which is then Morita equivalent with A. This process of constructing something new, given a special intermediate (or hybrid) structure, will appear again and again in more complex settings.

1.2 Comonoidal Morita equivalence of Hopf algebras

1.2.1 Hopf algebras and weak Hopf algebras

Definition 1.2.1. *A* coalgebra (A, Δ_A) *consists of a k-vector space A and a k-linear map*

$$\Delta_A : A \to A \odot A,$$

called the comultiplication *or* coproduct, *such that*

$$(\Delta_A \otimes \iota_A)\Delta_A = (\iota_A \otimes \Delta_A)\Delta_A \qquad (\textit{coassociativity}).$$

It is called counital *if there exists a k-linear map*

$$\varepsilon_A : A \to k$$

such that

$$(\varepsilon_A \otimes \iota_A)\Delta_A = \iota_A = (\iota_A \otimes \varepsilon_A)\Delta_A.$$

Such a map is then called a counit*.*

When doing calculations with the comultiplication, there is a convenient notation at hand, called the *Sweedler notation*, similar to the 'dot'-notation replacing the map M_A in an algebra. Namely, if $a \in A$, one writes

$$\Delta_A(a) = a_{(1)} \otimes a_{(2)},$$

the latter being just a formal expression encoding a sum of elementary tensors. This works of course for any map $\Delta_A : A \to A \odot A$. However, if Δ_A is coassociative, we can then write

$$(\Delta_A \otimes \iota_A)\Delta_A(a) = a_{(1)(1)} \otimes a_{(1)(2)} \otimes a_{(2)}$$

unambiguously as

$$\Delta_A^{(2)}(a) = a_{(1)} \otimes a_{(2)} \otimes a_{(3)},$$

since it then equals the (only) other possible intepretation

$$(\iota_A \otimes \Delta_A)\Delta_A(a) = a_{(1)} \otimes a_{(2)(1)} \otimes a_{(2)(2)}.$$

Remark that, just as the unit in a unital algebra is uniquely determined, also a counit ε_A of a counital coalgebra is uniquely determined, so we can talk about *the* counit.

Definition 1.2.2. *Let (A, Δ_A) be a (counital) coalgebra. The* opposite coalgebra *$(A^{cop}, \Delta_{A^{cop}})$ is the (counital) coalgebra $(A, \Sigma_{A,A} \circ \Delta_A)$. We also write $\Delta_{A^{cop}}$ as Δ_A^{op}.*

Definition 1.2.3. *A* bialgebra[2] *(A, M_A, Δ_A) consists of a unital algebra structure (A, M_A) and a counital coalgebra structure (A, Δ_A) on A, such that Δ_A and ε_A are unital homomorphisms of k-algebras.*

Definition 1.2.4. *A* Hopf algebra *(A, M_A, Δ_A) is a bialgebra for which there exists a bijective map $S_A : A \to A$, called* an antipode*, such that*

$$M_A(\iota_A \otimes S_A)\Delta_A = \eta_A\varepsilon_A = M_A(S_A \otimes \iota_A)\Delta_A.$$

[2]The terminology unital counital bialgebra would be more precise, but we refrain from using these extra specifications.

In Sweedler notation, this defining property becomes

$$S_A(a_{(1)})a_{(2)} = \varepsilon_A(a)1_A = a_{(1)}S_A(a_{(2)}).$$

We remark that such an antipode, when it exists, is unique, so we can talk about *the* antipode. We also remark that one often does not ask that S_A is bijective, but this will be the only case we are interested in. One can show that the antipode S_A is automatically an anti-multiplicative anti-comultiplicative map, the latter meaning that $\Delta_A \circ S_A = (S_A \otimes S_A)\Delta_A^{\mathrm{op}}$. The bijectivity of the antipode also implies that $(A, M_A, \Delta_A^{\mathrm{op}})$ and $(A, M_A^{\mathrm{op}}, \Delta_A)$, which we abbreviate again respectively as A^{cop} and A^{op}, are Hopf algebras, with antipode $S_{A^{\mathrm{cop}}} = S_{A^{\mathrm{op}}} = S_A^{-1}$.

We now give a different characterization of Hopf algebras, which will reappear in a generalized form from time to time.

Proposition 1.2.5. *Let (A, M_A) be a unital algebra, and (A, Δ_A) a coalgebra structure on A. Assume that Δ_A is a unital homomorphism. Then (A, M_A, Δ_A) is a Hopf algebra iff the maps*

$$T_{1,\Delta_A} : A \odot A \to A \odot A : a \otimes a' \to (a \otimes 1_A)\Delta_A(a'),$$

$$T_{2,\Delta_A} : A \odot A \to A \odot A : a \otimes a' \to (1_A \otimes a)\Delta_A(a'),$$

which are called the Galois maps *associated with Δ_A, are both bijections.*

Remark: Since (A, Δ_A) is a Hopf algebra iff $(A^{\mathrm{op}}, \Delta_A)$ is a Hopf algebra, we may also replace the maps T_{1,Δ_A} and T_{2,Δ_A} in the previous proposition by the maps

$$T_{\Delta_A,2} : A \odot A \to A \odot A : a \otimes a' \to \Delta_A(a)(1_A \otimes a'),$$

$$T_{\Delta_A,1} : A \odot A \to A \odot A : a \otimes a' \to \Delta_A(a)(a' \otimes 1_A).$$

Proof. Let (A, M_A, Δ_A) be a Hopf algebra. Define

$$T_{1,\Delta_A}^{-1} : A \odot A \to A \odot A : a \otimes a' \to aS_A(a'_{(1)}) \otimes a'_{(2)}.$$

Then an easy computation shows that this is an inverse for T_{1,Δ_A}. For example,

$$\begin{aligned}
T^{-1}_{1,\Delta_A}T_{1,\Delta_A}(a\otimes a') &= T^{-1}_{1,\Delta_A}(aa'_{(1)}\otimes a'_{(2)})\\
&= aa'_{(1)}S_A(a'_{(2)(1)})\otimes a'_{(2)(2)}\\
&= aa'_{(1)(1)}S_A(a'_{(1)(2)})\otimes a'_{(2)}\\
&= a\otimes \varepsilon_A(a'_{(1)})a'_{(2)}\\
&= a\otimes a'.
\end{aligned}$$

Similarly, the inverse of T_{2,Δ_A} is given as

$$T^{-1}_{2,\Delta_A}(a\otimes a') = a'S_A^{-1}(a_{(2)})\otimes a_{(1)}.$$

As for the converse statement, we refer to the proof of Lemma 1.2.18: simply replace B and D there by A to obtain that A has a counit and an antipode. The bijectivity of the antipode is not established there, but follows easily by the following argument: since A^{cop} also has bijective Galois maps, it has an antipode $S_{A^{\mathrm{cop}}}$. An easy argument shows that this is then an inverse for S_A (see for example the end of Lemma 1.2.15).

□

We will need the following simple lemma at one point.

Lemma 1.2.6. *Let A be a Hopf algebra, and I a right ideal of A. Suppose $\Delta_A(I)\subseteq I\odot I$. Then $I=0$ or $I=A$.*

Proof. Consider the restriction of the map

$$T_{\Delta_A,2}: A\odot A\to A\odot A: a\otimes a'\to \Delta_A(a)(1_A\otimes a')$$

to $I\odot A$. Then we can see it as a map T_I from $I\odot A$ to $I\odot A$, since $\Delta_A(I)\subseteq I\odot I$. Since the inverse of $T_{\Delta_A,2}$ is given as

$$T^{-1}_{\Delta_A,2}(a\otimes a') = a_{(1)}\otimes S_A(a_{(2)})a',$$

T_I is an invertible map. Now since I is a right ideal, we have in fact that the range of T_I ends up in $I\odot I$, hence we have $I\odot I = I\odot A$. Applying ε_I to the first leg, we see that either $I=A$, or $\varepsilon_A(I)=0$. But since $\Delta_A(I)\subseteq I\odot I$, the latter means $I=0$. □

We also introduce the notion of a *weak* bialgebra and *weak* Hopf algebra ([11]).

Definition 1.2.7. *A* weak bialgebra (E, M_E, Δ_E) *consists of a unital algebra* (E, M_E) *and counital coalgebra* (E, Δ_E)*, such that* Δ_E *is a homomorphism.*

A weak bialgebra is called monoidal *if* ε_E *is 'weakly multiplicative': for all* $x, y, z \in E$*, we have*

$$\varepsilon_E(xy_{(1)})\varepsilon_E(y_{(2)}z) = \varepsilon_E(xyz) = \varepsilon_E(xy_{(2)})\varepsilon_E(y_{(1)}z).$$

A weak bialgebra is called comonoidal *if the unit is 'weakly comultiplicative':*

$$\begin{aligned}\Delta_E^{(2)}(1_E) &= (\Delta_E(1_E) \otimes 1_E)(1_E \otimes \Delta_E(1_E)) \\ &= (1_E \otimes \Delta_E(1_E))(\Delta_E(1_E) \otimes 1_E).\end{aligned}$$

A weak Hopf algebra (E, M_E, Δ_E) *is a monoidal and comonoidal weak bialgebra for which there exists an invertible map* $S_E : E \to E$*, called* the antipode*, such that*

$$S_E(x_{(1)})x_{(2)} = \varepsilon_E(x1_{(2)})1_{(1)},$$

$$x_{(1)}S_E(x_{(2)}) = \varepsilon_E(1_{(1)}x)1_{(2)},$$

and

$$S_E(x_{(1)})x_{(2)}S_E(x_{(3)}) = S_E(x).$$

Again, the antipode is then automatically unique, and moreover anti-multiplicative, even without the bijectivity assumption on S_E (see [11]).

Associated to any weak Hopf algebra E, there are two natural unital subalgebras, called the *counital subalgebras*, which are anti-isomorphic to each other. One defines them as

$$E^t := \{x \in E \mid \Delta_E(x) = (x \otimes 1_E)\Delta_E(1_E) = \Delta_E(1_E)(x \otimes 1_E)\},$$

which is called the *range* or *target* subalgebra, and

$$E^s := \{x \in E \mid \Delta_E(x) = (1_E \otimes x)\Delta_E(1_E) = \Delta_E(1_E)(1_E \otimes x)\},$$

which is called the *source* subalgebra. (We note that in [11], E^t is denoted E^L, while E^s is denoted E^R.) The mentioned anti-isomorphism is then provided by the antipode S_E. One further shows that E^s and E^t commute, that $\Delta_E(1_E) \in E^s \odot E^t$, and that E^s is the first, E^t the second leg of $\Delta_E(1_E)$ (meaning that every element of E^s can be written as a linear combination of elements of the form $(\iota_E \otimes \omega)\Delta_E(1_E)$, where the ω are linear functionals on E, and similarly for E^t).

In the following, we will denote E^t by the symbol L, and call it the *object algebra* or *basis* of the weak Hopf algebra. We then denote the identity map from L to E^t by t_E, and call it the *target map*, while we denote the map $L \to E^s : x \to S_E^{-1}(x)$ by s_E, and call it the *source map*. We also introduce the notations

$$\mathcal{E}_t : E \to E^t : x \to \varepsilon_E(1_{(1)}x)1_{(2)}$$

and

$$\mathcal{E}_s : E \to E^s : x \to \varepsilon_E(x1_{(2)})1_{(1)}.$$

Then $\mathcal{E}_t$ is left E^t-linear and $\mathcal{E}_s$ is right E^s-linear.

We remark that weak Hopf algebras are to be seen as the non-commutative versions of affine groupoid schemes on a finite set of objects. The motivation for this is Proposition 2.11 of [11], which states that L is a separable, hence semi-simple algebra (this fact holds also in the case where E is infinite-dimensional). In fact, there are several general definitions of 'non-commutative affine groupoid schemes' in the literature, but while the weak Hopf algebra theory can in all cases be seen as a 'special case', we should remark however that they are in one sense more *refined* than most of these general objects, in that distinct weak Hopf algebras can become equal when passing to a more general theory. This has to do with the lack of a unique antipode (or even lack of an antipode) in these general theories (by which, to be complete, we mean: the Hopf algebroid theory of [60], the slightly more general Hopf algebroid theory proposed in [13], or the still more general $\times_R$-Hopf algebra theory of [72]). On the other hand, in the other theories, one has the algebra L from the outset, provided with an embedding and anti-embedding inside the quantum groupoid. Hence the weak Hopf algebra picture does not see the actual embedding of L, which could be perturbed by an automorphism of L.

1.2.2 Monoidal equivalence of categories

We will need the notion of a strict monoidal (k-additive) category, of a (co)monoidal functor, and of a morphism between (co)monoidal functors. We note that while the more general notion of a monoidal category is important in some situations (most notably for the quantization of semi-simple

Lie groups), we will be able to get by without it.

Definition 1.2.8. *A* strict monoidal k-linear category $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}})$ *consists of a k-linear category $\mathcal{C}$, a k-bilinear functor $\otimes_{\mathcal{C}} : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and an object $\mathbb{1}_{\mathcal{C}} \in \mathcal{C}$, such that*

$$X \otimes_{\mathcal{C}} (Y \otimes_{\mathcal{C}} Z) = (X \otimes_{\mathcal{C}} Y) \otimes_{\mathcal{C}} Z \qquad \forall X, Y, Z \in Ob(\mathcal{C}),$$

$$\mathbb{1}_{\mathcal{C}} \otimes_{\mathcal{C}} X = X = X \otimes_{\mathcal{C}} \mathbb{1}_{\mathcal{C}} \qquad \forall X \in Ob(\mathcal{C}).$$

Definition 1.2.9. *Let $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, \mathbb{1}_{\mathcal{D}})$ be two strict monoidal categories.*

A weak monoidal functor (F, u, v) *from $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}})$ to $(\mathcal{D}, \otimes_{\mathcal{D}}, \mathbb{1}_{\mathcal{D}})$ consists of a k-additive functor $F : \mathcal{C} \to \mathcal{D}$, together with a natural transformation $u : \otimes_{\mathcal{D}} \circ (F \times F) \to F \circ \otimes_{\mathcal{C}}$ and a morphism $v : \mathbb{1}_{\mathcal{D}} \to F(\mathbb{1}_{\mathcal{C}})$, such that*

$$u_{X \otimes_{\mathcal{C}} Y, Z}(u_{X,Y} \otimes_{\mathcal{D}} \iota_{F(Z)}) = u_{X, Y \otimes_{\mathcal{C}} Z}(\iota_{F(X)} \otimes_{\mathcal{D}} u_{Y,Z}) \qquad \textit{(2-cocycle relation)},$$

$$u_{\mathbb{1}_{\mathcal{C}}, X} \circ (v \otimes_{\mathcal{D}} \iota_{F(X)}) = \iota_{F(X)},$$

$$u_{X, \mathbb{1}_{\mathcal{C}}} \circ (\iota_{F(X)} \otimes_{\mathcal{D}} v) = \iota_{F(X)},$$

for all $X, Y, Z \in Ob(\mathcal{C})$.

A weak comonoidal functor (F, u, v) *from $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}})$ to $(\mathcal{D}, \otimes_{\mathcal{D}}, \mathbb{1}_{\mathcal{D}})$ consists of a k-additive functor $F : \mathcal{C} \to \mathcal{D}$, together with a natural transformation $u : F \circ \otimes_{\mathcal{C}} \to \otimes_{\mathcal{D}} \circ (F \times F)$ and a morphism $v : F(\mathbb{1}_{\mathcal{C}}) \to \mathbb{1}_{\mathcal{D}}$, such that*

$$(u_{X,Y} \otimes_{\mathcal{D}} \iota_{F(Z)}) u_{X \otimes_{\mathcal{C}} Y, Z} = (\iota_{F(X)} \otimes_{\mathcal{D}} u_{Y,Z}) u_{X, Y \otimes_{\mathcal{C}} Z},$$

$$(v \otimes_{\mathcal{D}} \iota_{F(X)}) \circ u_{\mathbb{1}_{\mathcal{C}}, X} = \iota_{F(X)},$$

$$(\iota_{F(X)} \otimes_{\mathcal{D}} v) \circ u_{X, \mathbb{1}_{\mathcal{C}}} = \iota_{F(X)},$$

for all $X, Y, Z \in Ob(\mathcal{C})$.

A (co-)monoidal functor (F, u, v) *between $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, \mathbb{1}_{\mathcal{D}})$ is a weak (co-)monoidal functor for which u is a natural isomorphism and v is an isomorphism.*

Note that if (F, u, v) is a monoidal functor, then (F, u^{-1}, v^{-1}) is a weak comonoidal functor. Hence it is not really necessary to introduce separately the notion of a 'comonoidal functor'. However, we will still do so, since sometimes the comonoidal structure is the most natural one to consider.

We can compose two monoidal functors (G, u', v') and (F, u, v), resp. from $(\mathcal{D}, \otimes_\mathcal{D}, \mathbb{1}_\mathcal{D})$ to $(\mathcal{E}, \otimes_\mathcal{E}, \mathbb{1}_\mathcal{E})$ and from $(\mathcal{C}, \otimes_\mathcal{C}, \mathbb{1}_\mathcal{C})$ to $(\mathcal{D}, \otimes_\mathcal{D}, \mathbb{1}_\mathcal{D})$, and obtain then a monoidal functor $(G \circ F, G(u) \circ u'_{F(\,\cdot\,),F(\,\cdot\,)}, G(v) \circ v')$ from $(\mathcal{C}, \otimes_\mathcal{C}, \mathbb{1}_\mathcal{C})$ to $(\mathcal{E}, \otimes_\mathcal{E}, \mathbb{1}_\mathcal{E})$.

Definition 1.2.10. *Let $(\mathcal{C}, \otimes_\mathcal{C}, \mathbb{1}_\mathcal{C})$ and $(\mathcal{D}, \otimes_\mathcal{D}, \mathbb{1}_\mathcal{D})$ be two strict monoidal categories. A* monoidal equivalence *(F, u, v) from $(\mathcal{C}, \otimes_\mathcal{C}, \mathbb{1}_\mathcal{C})$ to $(\mathcal{D}, \otimes_\mathcal{D}, \mathbb{1}_\mathcal{D})$ is a monoidal functor whose underlying functor F is part of an equivalence of categories.*

One should be careful with this notion of monoidal equivalence, as it is not really symmetric: one would like to know something about the monoidality of the quasi-inverse of F also. This is taken care of by the following lemma:

Lemma 1.2.11. *([70], I.4.4) Let (F, u, v) be a monoidal equivalence between two strict monoidal categories $(\mathcal{C}, \otimes_\mathcal{C}, \mathbb{1}_\mathcal{C})$ and $(\mathcal{D}, \otimes_\mathcal{D}, \mathbb{1}_\mathcal{D})$. Then there exists a monoidal equivalence (G, u', v') from $(\mathcal{D}, \otimes_\mathcal{D}, \mathbb{1}_\mathcal{D})$ to $(\mathcal{C}, \otimes_\mathcal{C}, \mathbb{1}_\mathcal{C})$, such that G is a quasi-inverse for F with counit $\varepsilon : FG \to \iota_\mathcal{D}$ and unit $\eta : \iota_\mathcal{C} \to GF$, and such that*

$$\eta_{X \otimes_\mathcal{C} Y} = G(u_{X,Y}) \circ u'_{F(X),F(Y)} \circ (\eta_X \otimes_\mathcal{C} \eta_Y) \qquad \textit{for all } X, Y \in Ob(\mathcal{C}),$$

$$(\varepsilon_X \otimes_\mathcal{D} \varepsilon_Y) = \varepsilon_{X \otimes_\mathcal{D} Y} \circ F(u'_{X,Y}) \circ u_{G(X),G(Y)} \qquad \textit{for all } X, Y \in Ob(\mathcal{D}).$$

We do not give a proof of this, but only note how u' is constructed:

$$\begin{array}{ccc} G(X) \otimes G(Y) & \xrightarrow{u'_{X,Y}} & G(X \otimes Y) \\ \downarrow{\scriptstyle \eta_{G(X)\otimes G(Y)}} & & \uparrow{\scriptstyle G(\varepsilon_X \otimes \varepsilon_Y)} \\ GF(G(X) \otimes G(Y)) & \xrightarrow[G(u^{-1}_{G(X),G(Y)})]{} & G((FG)(X) \otimes (FG)(Y)) \end{array} \tag{1.1}$$

Since the composition of monoidal equivalences produces a monoidal equivalence, this then shows that we really obtain an equivalence relation on the collection of strict monoidal categories.

We note that for a monoidal equivalence, one does not have to ask that v exists from the outset: it comes for free, in a canonical way ('a unit is automatically preserved under an algebra isomorphism'). Hence we may remove it from the data.

We will also talk about comonoidal equivalences, although, again, they contain the same information as monoidal equivalences.

Finally, we give the definition of a morphism between two monoidal functors:

Definition 1.2.12. *Let $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, \mathbb{1}_{\mathcal{D}})$ be two strict monoidal categories, and (F, u, v) and (F', u', v') two (weak) monoidal functors from $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}})$ to $(\mathcal{D}, \otimes_{\mathcal{D}}, \mathbb{1}_{\mathcal{D}})$. Then a* monoidal natural transformation *ϕ from (F, u, v) to (F', u', v') is a natural transformation $\phi : F \to F'$ such that*

$$u'_{X,Y} \circ (\phi_X \underset{\mathcal{D}}{\otimes} \phi_Y) = \phi_{X \underset{\mathcal{C}}{\otimes} Y} \circ u_{X,Y}, \qquad \textit{for all } X, Y \in Ob(\mathcal{C}),$$

and such that

$$\phi_{\mathbb{1}_{\mathcal{C}}} \circ v = v'.$$

It is clear then what is meant by a monoidal natural isomorphism between two monoidal functors.

1.2.3 Comonoidal Morita equivalence

We now associate to any weak Hopf algebra E a strict monoidal category. Let V and W be two unital left E-modules. We can make a new left unital E-module on the vector space $\Delta_E(1_E) \cdot (V \odot W)$ by defining

$$x \cdot (v \otimes w) := (x_{(1)} \cdot v) \otimes (x_{(2)} \cdot w).$$

In fact, since $(S_E \otimes \iota_E)\Delta_E(1)$ is a 'separating idempotent'[3] for L (cf. Proposition 2.12 of [11]), we can also canonically identify $\Delta_E(1_E) \cdot (V \odot W)$ with $V \underset{L}{\odot} W$ as a vector space by the natural projection map, where V is a right L-module by the anti-representation $\pi_V \circ s_E$, and W a left L-module by $\pi_W \circ t_E$. By a good choice for the universal construction of the (balanced) tensor product (cf. [73]), we may assume that this tensor product is strictly associative. Now denoting for $x \in E$ by $\pi_t(x)$ the map

$$\pi_t(x) : L \to L : l \to t_E^{-1}(\mathcal{E}_t(x t_E(l))),$$

[3] That is, writing $(S_E \otimes \iota_E)\Delta_E(1) = \sum_i p_i \otimes q_i$, we have $\sum_i p_i q_i = 1_E$ and $\sum_i l p_i \otimes q_i = \sum_i p_i \otimes q_i l$ for all $l \in L$

one can check that π_t is a left representation of E on L, which moreover provides a unit for the tensor product $\underset{L}{\odot}$, since we have for any left E-module V that $V \underset{L}{\odot} L = V = L \underset{L}{\odot} V$, with strict equality again when the balanced tensor product is appropriately defined. Then $(E\text{-Mod}, \underset{L}{\odot}, \pi_t)$ becomes a strict monoidal category. When we regard left modules as left representations, we will denote the tensor product by $\underset{L}{\cdot}$, so $\pi_1 \underset{L}{\cdot} \pi_2$ is the 'tensor product representation' associated with the representations π_1 and π_2. When E is in fact an ordinary Hopf algebra A, we use the same notation, but simply delete the symbol L everywhere, while π_t then becomes the trivial representation ε_A.

In the same way, we can turn the category of unital right modules of a weak Hopf algebra E into a strict monoidal category $(\text{Mod-}E, \underset{L}{\odot}, \pi_s)$.

We can now define the following natural concept.

Definition 1.2.13. *Let A and D be two Hopf algebras. We call them* comonoidally Morita equivalent *if the monoidal categories $(\text{Mod-}A, \odot, \varepsilon_A)$ and $(\text{Mod-}D, \odot, \varepsilon_D)$ are comonoidally equivalent. We call a particular such comonoidal equivalence a* comonoidal Morita equivalence *between the two Hopf algebras.*

The motivation for calling this a *comonoidal* equivalence will be given after Proposition 1.2.17.

Our aim is again to recapture this notion in a more concrete way.

Definition 1.2.14. *A* linking weak Hopf algebra *consists of a unital linking algebra (E, e), where E is equipped with the structure of a weak Hopf algebra in such a way that*

$$\Delta_E(e) = e \otimes e$$

and

$$\Delta_E(1_E - e) = (1_E - e) \otimes (1_E - e).$$

If A and D are two Hopf algebras, we call a quadruple (E, e, Φ_A, Φ_D) consisting of a linking weak Hopf algebra (E, e) and comultiplication preserving isomorphisms

$$\Phi_A : (A, \Delta_A) \to (eEe, (\Delta_E)_{|eEe}),$$

$$\Phi_D : (D, \Delta_D) \to ((1_E - e)E(1_E - e), (\Delta_E)_{|(1_E-e)E(1_E-e)})$$

a linking weak Hopf algebra between A *and* D*.*

We will apply the same conventions as for unital linking algebras, so we actually do not explicitly write down the identifying isomorphisms Φ_A and Φ_D.

As remarked at the end of the subsection on Hopf and weak Hopf algebras, any weak Hopf algebra E comes together with another algebra L, embedded in E in two ways. Since we have also remarked there that the counital subalgebras of E, i.e. the images of L under these embeddings, are exactly the left and right legs of $\Delta_E(1_E)$, it is easy to see that, in the case of a linking weak Hopf algebra, we have $L = E^t = E^s \cong k^2$, where the identification sends the canonical basis vector e_1 of k^2 to $1_E - e$, and the basis vector e_2 to e.

This allows us to view linking weak Hopf algebras in the following way: they can be seen as the 'groupoid algebra' pertaining to a quantum groupoid with a *classical* object space consisting of two points, with A and D playing the rôle of the group algebra of the endomorphism groups of the two points, and with B and C playing the rôle of 'arrow bimodules' for the set of morphisms between the two objects. Composition of morphisms then corresponds to the algebra multiplications and bimodule structures.

Lemma 1.2.15. *Let* A *and* D *be two Hopf algebras. Let* (E, e) *be a linking algebra between the algebras underlying* A *and* D*, and suppose* Δ_E *is a coassociative homomorphism* $E \to E \odot E$*, such that* $(\Delta_E)_{|A} = \Delta_A$ *and* $(\Delta_E)_{|D} = \Delta_D$*. Then* (E, e) *is a linking weak Hopf algebra between* A *and* D*.*

Proof. By the assumptions, we have that $\Delta_E(e) = e \otimes e$ and $\Delta_E(1_E - e) = (1_E - e) \otimes (1_E - e)$. We have to show that E possesses a counit and antipode. We will write $E = \begin{pmatrix} D & B \\ C & A \end{pmatrix}$ as before, and we denote by Δ_B and Δ_C the restrictions of Δ_E to resp. B and C.

We first note that the map

$$T_{1,\Delta_B} : D \odot B \to B \odot B : d \otimes b \to (d \otimes 1)\Delta_B(b)$$

is a bijection. Indeed: suppose for example that $b_i \in B$ and $d_i \in D$ are such that $\sum_i (d_i \otimes 1)\Delta_B(b_i) = 0$. Then for any $c \in C$, we have

$$\begin{aligned} \sum_i (d_i \otimes 1)\Delta_B(b_i)\Delta_C(c) &= \sum_i (d_i \otimes 1)\Delta_D(b_i c) \\ &= 0, \end{aligned}$$

so that $\sum d_i \otimes b_i c = 0$ by Proposition 1.2.5. Since c was arbitrary, and $C \cdot B = A$, we conclude $\sum_i d_i \otimes b_i = 0$. Hence T_{1,Δ_B} is injective. On the other hand, choose $b_i \in B$ and $c_i \in C$ with $\sum_i c_i b_i = 1_A$. Choose $b, b' \in B$ and write $(b \otimes b')\Delta_C(c_i) = \sum_j (q_{ij} \otimes 1_D)\Delta_D(p_{ij})$ for certain $p_{ij}, q_{ij} \in D$. Then

$$\begin{aligned} b \otimes b' &= \sum_i (b \otimes b')\Delta_C(c_i)\Delta_B(b_i) \\ &= \sum_{i,j} (q_{ij} \otimes 1_D)\Delta_B(p_{ij} b_i), \end{aligned}$$

so that T_{1,Δ_B} is also surjective, hence bijective.

Then the beginning of the proof of Proposition 1.2.18 lets us conclude that (B, Δ_B) is in fact a counital coalgebra, i.e. possesses a counit ε_B, and that

$$\varepsilon_B(db) = \varepsilon_D(d)\varepsilon_B(b).$$

By symmetry (interchanging e and $1_E - e$), we have that (C, Δ_C) is a counital coalgebra, with counit ε_C. Symmetry (interchanging the multiplication in E and the opposite multiplication), and the uniqueness of a counit, also lets us conclude that $\varepsilon_B(d \cdot b \cdot a) = \varepsilon_D(d)\varepsilon_B(b)\varepsilon_A(a)$, and then also $\varepsilon_C(a \cdot c \cdot d) = \varepsilon_A(a)\varepsilon_C(c)\varepsilon_D(d)$ for $d \in D, b \in B$ and $a \in A$. A similar argument as the one showing that $\varepsilon_B(db) = \varepsilon_D(d)\varepsilon_B(b)$ also let us conclude that $\varepsilon_D(bc) = \varepsilon_B(b)\varepsilon_C(c)$ and that $\varepsilon_A(cb) = \varepsilon_C(c)\varepsilon_B(b)$ for $b \in B$ and $c \in C$.

Put

$$\varepsilon_E(\begin{pmatrix} d & b \\ c & a \end{pmatrix}) := \varepsilon_D(d) + \varepsilon_C(c) + \varepsilon_B(b) + \varepsilon_A(a).$$

Then (E, M_E, Δ_E) is a comonoidal and monoidal weak bialgebra. In fact, it is immediate that ε_E is a counit for Δ_E, since $(E, \Delta_E, \varepsilon_E)$ is just the direct sum coalgebra of the coalgebras A, B, C and D. The weak multiplicativity of ε_E follows easily from the bimodularity of its constituents, while the weak comultiplicativity of $\Delta_E(1_E) = (e \otimes e) + (1_E - e) \otimes (1_E - e)$ is immediate.

Now we show that (E, M_E, Δ_E) is a weak *Hopf* algebra, i.e., that there exists an antipode S_E. Again, as in the proof of Proposition 1.2.18, we can construct a map $S_B : B \to C \cong \mathrm{Hom}_D({}_D B, {}_D D)$ such that $S_B(b_{(1)})b_{(2)} = \varepsilon_B(b)$ and $b_{(1)} S_B(b_{(2)}) = \varepsilon_B(b)$. By symmetry, we can also construct a map $S_C : C \to B$, satisfying similar conditions. Then one easily verifies that $S_E := S_D \oplus S_C \oplus S_B \oplus S_A$ satisfies the conditions for an antipode on (E, Δ_E).

We still have to show that S_E is bijective, or, which is the same, that S_B and S_C are bijective. By symmetry, it is enough to check this for S_B. But by considering the comultiplication Δ_E^{op} on E, which satisfies the assumptions of the lemma with respect to A^{cop} and D^{cop}, we find a map $C \to B$, suggestively written as S_B^{-1} already, such that

$$S_B^{-1}(b_{(2)})b_{(1)} = \varepsilon_B(b)1_A.$$

Since it is known, by the general theory of weak Hopf algebras, that S_E is anti-multiplicative, we see then, applying S_A, that for all $b \in B$,

$$S_B(b_{(1)})(S_B S_B^{-1})(b_{(2)}) = \varepsilon_B(b)1_A.$$

Then

$$\begin{aligned}(S_B S_B^{-1})(b) &= b_{(1)}S_B(b_{(2)})(S_B S_B^{-1})(b_{(3)}) \\ &= b_{(1)} \cdot (\varepsilon_B(b_{(2)})1_A) \\ &= b.\end{aligned}$$

A similar argument shows that $(S_B^{-1}S_B)(b) = b$, so that S_B^{-1} is really the inverse of S_B.

□

Remark: In fact, we do not even need to assume we have an underlying linking algebra from the start. For let E be an algebra, e a projection in E, and Δ_E a coassociative homomorphism $E \to E \odot E$ such that $\Delta_E(e) = e \otimes e$ and $\Delta_E(1_E - e) = (1_E - e) \otimes (1_E - e)$. Suppose further that $A = eEe$ and $D = (1_E - e)E(1_E - e)$, equipped with the restriction of Δ_E, are Hopf algebras. Suppose that, denoting $B = (1_E - e)Ee$ and $C = eE(1_E - e)$, either $B \cdot C$ or $C \cdot B \neq 0$. Then, for example when $B \cdot C$ is not zero, it is clearly a right ideal of D, satisfying $\Delta_D(B \cdot C) \subseteq (B \cdot C) \odot (B \cdot C)$. By Lemma 1.2.6, we then have $B \cdot C = D$. Since then $B = B \cdot (C \cdot B)$, also $C \cdot B \neq 0$, and a similar argument gives that $C \cdot B = A$. Hence E is a linking algebra between A and D.

Now we look again at the one-sided, asymmetric situation.

Definition 1.2.16. *Let D be a Hopf algebra. A left* comonoidal Morita D-module (B, Δ_B) *consists of a non-zero left Morita D-module B together with a coassociative left D-module map $\Delta_B : B \to B \odot B$, such that the map*

$$D \odot B \to B \odot B : d \otimes b \to (d \otimes 1)\Delta_B(b)$$

is an isomorphism.

If A is another Hopf algebra, then we call a triple (B, Δ_B, θ) consisting of a left comonoidal Morita D-module (B, Δ_B) and an anti-isomorphism $\theta : A \to End_D({}_DB)$ such that

$$\Delta_B((\theta(a))(b)) = (\theta(a_{(1)}) \otimes \theta(a_{(2)}))\Delta_B(b)$$

a comonoidal equivalence bimodule *between A and D.*

In fact, this definition is formulated too strongly, as we will show in the next subsection.

Remark: We note that the terminology of 'comonoidal Morita D-module' may not be too well-chosen: the bijectivity of the map, stated in the definition, should really be seen as an extra condition on the object which *should* be called a 'comonoidal Morita D-module'. The point is that these more general comonoidal Morita D-modules would then lead to equivalence bimodules between a Hopf algebra and some 'Hopf algebroid'. However, since we will not investigate this generalization, we will stick with the above terminology. We note that in the literature, one calls comonoidal Morita modules 'Galois coobjects'.

Proposition 1.2.17. *Let A and D be Hopf algebras. There is a one-to-one correspondence between isomorphism classes of*

1. *comonoidal Morita equivalences between A and D,*
2. *linking weak Hopf algebras between A and D, and*
3. *comonoidal equivalence bimodules between A and D.*

In particular, A and D are comonoidally Morita equivalent iff there exists a linking weak Hopf algebra between them, iff there exists a comonoidal equivalence bimodule between them.

Proof. Given either a comonoidal Morita equivalence, a linking weak Hopf algebra or a comonoidal equivalence bimodule between A and D, we have in particular respectively a Morita equivalence, unital linking algebra and equivalence bimodule. By Proposition 1.1.12, we know how to pass from one of these structures to the other. Our job is to show that the extra structure is carried along these correspondences.

Let (G, u) be a comonoidal Morita equivalence between D and A. Let $B = G(D_D)$ be the associated D-A-equivalence bimodule. We will give it the structure of a comonoidal D-A-equivalence bimodule.

In $(\text{Mod-}D, \odot, \varepsilon_D)$, we have the morphism

$$\Delta_D : D_D \to (D \odot D)_D.$$

Denote by Δ_B the morphism

$$u_{D,D} \circ G(\Delta_D) : B \to B \odot B.$$

Then Δ_B is coassociative:

$$\begin{aligned}
(\iota_B \otimes \Delta_B)\Delta_B &= (\iota_B \otimes (u_{D,D} \circ G(\Delta_D))) \circ (u_{D,D} \circ G(\Delta_D)) \\
&\underset{\text{naturality}}{=} (\iota_B \otimes u_{D,D}) \circ u_{D,D\odot D} \circ G(\iota_D \otimes \Delta_D) \circ G(\Delta_D) \\
&\underset{\text{2-cocycle id.}}{=} (u_{D,D} \otimes \iota_B) \circ u_{D\odot D,D} \circ G(\Delta_D \otimes \iota_D) \circ G(\Delta_D) \\
&= \ldots \\
&= (\Delta_B \otimes \iota_B)\Delta_B.
\end{aligned}$$

Now since Δ_B is a morphism of right A-modules, we must have

$$\Delta_B(b \cdot a) = \Delta_B(b) \cdot \Delta_A(a).$$

Further, denoting, for $d \in D$, by l_d the linear map 'left multiplication with d' in $\text{End}_D(D_D)$, we also have

$$\begin{aligned}
\Delta_B \circ G(l_d) &= u_{D,D} \circ G(\Delta_D) \circ G(l_d) \\
&= u_{D,D} \circ G(l_{d_{(1)}} \otimes l_{d_{(2)}}) \circ G(\Delta_D) \\
&\underset{\text{naturality}}{=} (G(l_{d_{(1)}}) \otimes G(l_{d_{(2)}})) \circ u_{D,D} \circ G(\Delta_D).
\end{aligned}$$

Hence

$$\Delta_B(d \cdot b) = \Delta_D(d) \cdot \Delta_B(b),$$

by definition of the left D-module structure on B.

We want to show that

$$D \odot B \to B \odot B : d \otimes b \to (d \otimes 1)\Delta_B(b)$$

is bijective. Let $\{e_i\}_{i\in I}$ be a basis of D. Denote by T_{1,Δ_D} the D-module morphism

$$T_{1,\Delta_D} : (D \odot (D_D) \cong) \bigoplus_{i\in I} D_D \to (D \odot D)_D :$$

$$(\sum_i e_i \otimes d_i \cong) \oplus d_i \to \sum_i (e_i \otimes 1)\Delta_D(d_i).$$

We know by Proposition 1.2.5 that T_{1,Δ_D} is bijective. Now we can write

$$T_{1,\Delta_D} = \bigoplus_{i\in I} (l_{e_i} \otimes 1) \circ \Delta_D,$$

where l_{e_i} is again left multiplication with the element e_i. So also

$$u_{D,D} \circ G(T_{1,\Delta_D})$$

is bijective. But G preserves direct sums, so

$$\begin{aligned} u_{D,D} \circ G(T_{1,\Delta_D}) &\cong \bigoplus_i (u_{D,D} \circ G(l_{e_i} \otimes 1) \circ G(\Delta_D)) \\ &= \bigoplus_i ((G(l_{e_i}) \otimes 1) \circ u_{D,D} \circ G(\Delta_D)), \end{aligned}$$

which says exactly that

$$D \odot B (\cong \bigoplus_i B) \to B \odot B : \sum_i e_i \otimes b_i (\cong \oplus b_i) \to \sum_i (e_i \otimes 1)\Delta_B(b_i)$$

is bijective. Hence B is a comonoidal D-A-equivalence bimodule.

(We also would like to present the construction of the coproduct in the case where we identify B with $\mathscr{B} = \mathrm{Hom}(U_D, U_A \circ G)$, where U denoted the forgetful functor to the category of vector spaces over k. Denote for the moment $(\mathcal{D}, \otimes_{\mathcal{D}}) = (\text{Mod-}D, \odot)$ and $(\mathcal{A}, \otimes_{\mathcal{A}}) = (\text{Mod-}A, \odot)$. First remark that we have a natural map $\Delta_{\mathscr{B}}$ from $\mathscr{B}$ to $\mathrm{Hom}(U_D \circ \otimes_{\mathcal{D}}, U_A \circ G \circ \otimes_{\mathcal{A}})$, by putting $\Delta_{\mathscr{B}}(b)_{(V,W)} := b_{V\otimes_{\mathcal{D}} W}$ for $b \in \mathscr{B}$. By composing with u, and noting that U intertwines the tensor products of $\mathcal{A}$ and $\mathcal{D}$ with the one of Mod-k, we can see $\Delta_{\mathscr{B}}$ as an element of $\mathrm{Hom}(\underset{k}{\odot} \circ (U_D \times U_D), \underset{k}{\odot} \circ ((U_A \circ G) \times (U_A \circ G)))$. But it is not difficult to show that this last space is isomorphic to $\mathscr{B} \odot \mathscr{B}$, where $b \otimes b'$ corresponds to the natural transformation $(b \otimes b')_{V \otimes_{\mathcal{D}} W} := b_V \otimes b'_V$. Hence $\Delta_{\mathscr{B}}$ can be interpreted as a map $\mathscr{B} \to \mathscr{B} \odot \mathscr{B}$. One then argues that it satisfies all needed properties. It is also easily seen to correspond exactly to the comultiplication on B, simply by evaluating elements of $\mathscr{B}$ at D.)

Next, suppose that B is a comonoidal D-A-equivalence bimodule. Let (E, e) be the linking algebra between A and D associated to the underlying equivalence bimodule. We write again $C = (1_E - e)Ee$, and we identify it with $\mathrm{Hom}_D({}_DB, {}_DD)$ in the natural way, by right multiplication. Since $\begin{pmatrix} D \odot D & B \odot B \\ C \odot C & A \odot A \end{pmatrix}$ is a linking algebra between $A \odot A$ and $D \odot D$, we can also identify $C \odot C$ with $\mathrm{Hom}_{D\odot D}({}_{(D\odot D)}(B \odot B), {}_{(D\odot D)}(D \odot D))$, where we again write the action of $C \odot C$ on $B \odot B$ as right multiplication.

By definition of a comonoidal D-A-equivalence bimodule, we know that

$$D \odot B \to B \odot B : d \otimes b \to (d \otimes 1)\Delta_B(b)$$

is bijective. But then also

$$B \odot D \to B \odot B : b \otimes d \to (1 \otimes d)\Delta_B(b)$$

is bijective, since

$$\begin{aligned} (1 \otimes d)\Delta_B(b) &= (S_D(d_{(1)})d_{(2)} \otimes d_{(3)})\Delta_B(b) \\ &= (S_D(d_{(1)}) \otimes 1)\Delta_B(d_{(2)}b), \end{aligned}$$

and

$$D \odot B \to D \odot B : d \otimes b \to S_D(d_{(1)}) \otimes d_{(2)}b$$

is bijective with

$$D \odot B \to D \odot B : d \otimes b \to S_D^{-1}(d_{(2)}) \otimes d_{(1)}b$$

as inverse.

Now take $c \in C$, and write

$$\phi_c : B \odot B \to D \odot D : \phi_c((d \otimes 1)\Delta_B(b)) = (d \otimes 1)\Delta_D(b \cdot c).$$

By the definition of a comonoidal equivalence bimodule, this is well-defined. Now for $d' \in D$, we trivially have

$$\phi_c((d'd \otimes 1)\Delta_B(b)) = (d' \otimes 1)\phi_c((d \otimes 1)\Delta_B(b)).$$

On the other hand, writing $1 \otimes d = S_D(d_{(1)})d_{(2)} \otimes d_{(3)}$, we also have

$$\begin{aligned} \phi_c((1 \otimes d)\Delta_B(b)) &= \phi_c((S_D(d_{(1)})d_{(2)} \otimes d_{(3)})\Delta_B(b)) \\ &= \phi_c((S_D(d_{(1)}) \otimes 1)\Delta_B(d_{(2)} \cdot b)) \\ &= (S_D(d_{(1)}) \otimes 1)\Delta_D(d_{(2)} \cdot b \cdot c) \\ &= (S_D(d_{(1)})d_{(2)} \otimes d_{(3)}))\Delta_D(b \cdot c) \\ &= (1 \otimes d)\Delta_D(b \cdot c), \end{aligned}$$

which, by the previous paragraph, gives an equivalent defining equality for ϕ_c. This makes it clear that also

$$\phi_c((1\otimes d'd)\Delta_B(b)) = (1\otimes d')\phi_c((1\otimes d)\Delta_B(b))$$

for $d' \in D$. By the discussion before the previous paragraph, this means that we can write

$$\phi_c(b\otimes b') = (b\otimes b')\Delta_C(c),$$

for a uniquely determined element $\Delta_C(c) \in C\odot C$.

It is easily verified that Δ_C is coassociative: using Sweedler notation, we have

$$\begin{aligned}
&(d\otimes d'\otimes d'')\Delta_B^{(2)}(b)((\iota_C\otimes\Delta_C)\Delta_C(c))\\
&\quad= db_{(1)}c_{(1)}\otimes d'b_{(2)(1)}c_{(2)(1)}\otimes d''b_{(2)(2)}c_{(2)(2)}\\
&\quad= db_{(1)}c_{(1)}\otimes d'(b_{(2)}c_{(2)})_{(1)}\otimes d''(b_{(2)}c_{(2)})_{(2)}\\
&\quad= d(bc)_{(1)}\otimes d'(bc)_{(2)(1)}\otimes d''(bc)_{(2)(2)}\\
&\quad= d(bc)_{(1)(1)}\otimes d'(bc)_{(1)(2)}\otimes d''(bc)_{(2)}\\
&\quad= d(b_{(1)}c_{(1)})_{(1)}\otimes d'(b_{(1)}c_{(1)})_{(2)}\otimes d''(b_{(2)}c_{(2)})\\
&\quad= db_{(1)(1)}c_{(1)(1)}\otimes d'b_{(1)(2)}c_{(1)(2)}\otimes d''b_{(2)}c_{(2)}\\
&\quad= (d\otimes d'\otimes d'')\Delta_B^{(2)}(b)((\Delta_C\otimes\iota_C)\Delta_C(c)),
\end{aligned}$$

which is sufficient to conclude $(\iota_C\otimes\Delta_C)\Delta_C(c) = (\Delta_C\otimes\iota_C)\Delta_C(c)$, since elements of the form $(d\otimes d'\otimes d'')\Delta_B^{(2)}(b)$ span $B\odot B\odot B$, on which $C\odot C\odot C$ acts faithfully by right multiplication.

We can now take the direct sum Δ_E of the $\Delta_D, \Delta_C, \Delta_B$ and Δ_A, and see this direct sum as a map $E\to E\odot E$, by embedding $D\odot D, C\odot C, B\odot B$ and $A\odot A$ in $E\odot E$ in the natural way. Then clearly, Δ_E is coassociative. We want to show that it is also multiplicative.

Now Δ_A and Δ_D are multiplicative on resp. A and D, by definition. Also, by definition,

$$\Delta_B(d\cdot b\cdot a) = \Delta_D(d)\cdot\Delta_B(b)\cdot\Delta_A(a),$$

and $\Delta_B(b)\Delta_C(c) = \Delta_D(bc)$. We also have that

$$\begin{aligned}
(d\otimes 1)\Delta_B(b)\Delta_C(c)\Delta_B(b') &= (d\otimes 1)\Delta_D(bc)\Delta_B(b')\\
&= (d\otimes 1)\Delta_B(bcb')\\
&= (d\otimes 1)\Delta_B(b)\Delta_A(cb'),
\end{aligned}$$

hence $\Delta_C(c)\Delta_B(b) = \Delta_A(cb)$. Similarly, one proves that $\Delta_C(a \cdot c \cdot d) = \Delta_A(a) \cdot \Delta_C(c) \cdot \Delta_D(d)$. All this combined proves the multiplicativity of Δ_E.

By Lemma 1.2.15, (E, e) is a linking weak Hopf algebra between A and D.

Now suppose that (E, e) is a linking weak Hopf algebra between A and D. By symmetry, we only have to construct a comonoidal equivalence from Mod-E to Mod-A. However, the restriction functor from Mod-E to Mod-A is already *strictly* comonoidal, i.e. $\mathrm{Res}(V) \odot \mathrm{Res}(W) = \mathrm{Res}(V \underset{k^2}{\odot} W)$, as is easily verified. (Of course, we have to choose the proper (balanced) tensor product of the vector spaces to have *equality* of the tensor product and the restriction of the balanced tensor product, but this can easily be achieved). It is also easily verified that the comonoidal structure u, obtained on the associated equivalence $- \underset{D}{\odot} B$ from Mod-D to Mod-A, equals

$$u_{V,W}(b \underset{D}{\otimes} (v \otimes w)) = (b_{(1)} \underset{D}{\otimes} v) \otimes (b_{(2)} \underset{D}{\otimes} w).$$

We again have to show that these constructions, when applied successively, give us back the original structure, up to isomorphism.

So suppose we start with a comonoidal Morita equivalence (F, u) between (Mod-$A, \odot, \varepsilon_A$) and (Mod-$D, \odot, \varepsilon_D$), and let (G, u') be a comonoidal quasi-inverse. In Proposition 1.1.12, we then constructed an isomorphism ϕ between the functors G and $\tilde{G} = - \underset{D}{\odot} B$, with $B = G(D_D)$. Now if we denote the comonoidal structure that we get on $\tilde{G}$ by $\tilde{u}$, we should show that ϕ changes $\tilde{u}'$ into u'. By construction of ϕ, this reduces to proving

$$(u'_{V,W} \circ G(l_{v \otimes w}))(b) = G(l_v)(b_{(1)}) \otimes G(l_w)(b_{(2)}),$$

where l was the operation introduced in Proposition 1.1.12. Now it is easily seen that $l_{v \otimes w} = (l_v \otimes l_w) \circ \Delta_D$. By naturality of u', we get that

$$\begin{aligned} u'_{V,W} \circ G(l_{v \otimes w}) &= u'_{V,W} \circ G(l_v \otimes l_w) \circ G(\Delta_D) \\ &= (G(l_v) \otimes G(l_w)) \circ u'_{D,D} \circ G(\Delta_D) \\ &= (G(l_v) \otimes G(l_w)) \circ \Delta_B, \end{aligned}$$

by construction of Δ_B. This then proves the equality we were after.

Starting with a comonoidal equivalence bimodule B between A and D, it is, as in Proposition 1.1.12, more trivial to see that the constructions lead us back to B itself.

Since the comultiplication Δ_C on the C-part of a linking weak Hopf algebra (E, e) is uniquely determined by the Δ_B and Δ_D-part, by the formula

$$((d \otimes 1)\Delta_B(b))\Delta_C(c) = (d \otimes 1)\Delta_D(bc),$$

it also follows immediately by the proof of the corresponding statement in Proposition 1.1.12 that applying our constructions successively on a linking algebra between A and D, we are led back to an isomorphic linking algebra between A and D.

□

We can now explain why we have chosen for the terminology of *comonoidal* Morita equivalence. For it follows from the above proposition that if A and D are two comonoidally Morita equivalent Hopf algebras, then the functorial part of the comonoidal equivalence functor Mod-D → Mod-A is given by the right balanced tensor product functor

$$V \to V \underset{D}{\odot} B,$$

while the 'comonoidal part' is given by

$$(V \odot W) \underset{D}{\odot} B \to (V \underset{D}{\odot} B) \otimes (W \underset{D}{\odot} B),$$

$$(v \otimes w) \underset{D}{\otimes} b \to (v \underset{D}{\otimes} b_{(1)}) \otimes (w \underset{D}{\otimes} b_{(2)}).$$

But this last formula makes sense for *any* comonoidal D-A-bimodule, i.e. for any D-A-bimodule B which is also a coalgebra, and whose comultiplication is a bimodule map. So the comonoidal structure really seems the most natural one to consider in this context.

1.2.4 Reflecting across Morita module coalgebras

Just as for Morita modules, one can construct from a comonoidal Morita module a new Hopf algebra, built on its endomorphism algebra. Apart from this, the following proposition also shows, maybe more surprisingly, that the definition we gave for a comonoidal Morita module is too strongly formulated: one only needs the bijectivity statement in that definition, since the

Morita property comes for free.

Proposition 1.2.18. *Let D be a Hopf algebra, and B a left D-module. Suppose B is a coalgebra, such that $\Delta_B(d \cdot b) = \Delta_D(d) \cdot \Delta_B(b)$, and such that the map*

$$D \odot B \to B \odot B : d \otimes b \to (d \otimes 1)\Delta_B(b)$$

is an isomorphism. Then B is a left comonoidal Morita D-module, and there exists a Hopf algebra A which completes it to a comonoidal equivalence bimodule between A and D. It is unique in the following sense: if A_1 is another Hopf algebra, and B is also a comonoidal equivalence bimodule between A_1 and D, then there exists an isomorphism $\Phi : A \to A_1$ of Hopf algebras, such that $b \cdot \Phi(a) = b \cdot a$ for all $a \in A$ and $b \in B$.

Proof. We note first that also

$$D \odot B \to B \odot B : d \otimes b \to (1 \otimes d)\Delta_B(b)$$

is an isomorphism, since

$$(1 \otimes d)\Delta_B(b) = (S_D(d_{(1)}) \otimes 1)\Delta_B(d_{(2)}b).$$

(When we want to use the ensuing proof for Proposition 1.2.5, this was in fact an extra assumption.)

We first show that B is a counital coalgebra. The argument is in fact completely the same as the one of [92]. (To be able to reuse this proof for another proposition, we will insert at places some steps which are redundant for *this* particular proof.) Choose $b \in B$. We define a map $E_B(b) : B \to B$ as follows. For any $b' \in B$, write

$$b' \otimes b = \sum_i (d_i \otimes 1)\Delta_B(b''_i),$$

where the expression on the right is uniquely determined by assumption. Then define

$$(E_B(b))(b') := \sum_i d_i \cdot b''_i.$$

We want to show that $E_B(b)$ is in fact a scalar, i.e. of the form $\varepsilon_B(b) \cdot \iota_B$ for some $\varepsilon_B(b) \in k$. Now by an easy argument, we also have that if $\sum_i b'_i \otimes b_i = \sum_j (d_j \otimes 1)\Delta_B(b''_j)$, then $\sum_i (E_B(b_i))(b'_i) = \sum_j d_j \cdot b''_j$. In particular,

$(E_B(b_{(2)}))(db_{(1)}) = db$. Further, if $b' \otimes b = \sum_i (d_i \otimes 1)\Delta_B(b''_i)$, then also $b' \otimes b_{(1)} \otimes d'b_{(2)} = \sum_i (d_i \otimes 1 \otimes d')(b''_{i(1)} \otimes b''_{i(2)} \otimes b''_{i(3)})$. Hence

$$\begin{aligned}(E_B(b_{(1)}))(b') \otimes d'b_{(2)} &= \sum_i d_i b''_{i(1)} \otimes d' b''_{i(2)} \\ &= b' \otimes d'b,\end{aligned}$$

from which we conclude that $\omega(d'b_{(2)})E_B(b_{(1)}) = \omega(d'b)\iota_B$ for any $\omega \in B^*$. Since $B \odot B = (1 \otimes D) \cdot \Delta_B(B)$, we conclude from this that E_B has indeed range in $k \cdot \iota_B$, and we can write $E_B(b) = \varepsilon_B(b)\iota_B$. The last identity for E_B then lets us conclude that $\varepsilon_B(b_{(1)})d'b_{(2)} = d'b$, and since we had already derived that $db_{(1)}\varepsilon_B(b_{(2)}) = db$ earlier on, we conclude that ε_B is indeed a counit for the coalgebra B.

We now show that ε_B satisifies $\varepsilon_B(d \cdot b) = \varepsilon_D(d)\varepsilon_B(b)$ (for which we will not need that ε_D is multiplicative). Take $b, b' \in B$ and $d \in D$. Write $b' \otimes b = \sum_i d_i p_{i(1)} \otimes p_{i(2)}$, and write $d_i \otimes d = \sum_j d_{ij} d'_{ij(1)} \otimes d'_{ij(2)}$. Then $b' \otimes db = \sum_{i,j} d_{ij}(d'_{ij}p_i)_{(1)} \otimes (d'_{ij}p_i)_{(2)}$, and by the counit property of ε_D and ε_B, we find

$$\begin{aligned}\varepsilon_B(d \cdot b)b' &= \sum_{i,j} d_{ij} d'_{ij} p_i \\ &= \sum_i \varepsilon_D(d) \sum_i d_i p_i \\ &= \varepsilon_D(d)\varepsilon_B(b)b',\end{aligned}$$

from which $\varepsilon_B(d \cdot b) = \varepsilon_D(d)\varepsilon_B(b)$ follows.

Now define a map $S_B : B \to \mathrm{Hom}_k(B, D)$ by the defining property that

$$(S_B(b_{(2)}))(db_{(1)}) = \varepsilon_B(b)d.$$

In fact, since $(S_B(b_{(2)}))(d'db_{(1)}) = d'(S_B(b_{(2)}))(db_{(1)})$, we see that S_B has range in $C := \mathrm{Hom}_D({}_DB, {}_DD)$. *From now on, we let elements of C act on the right of B*, so in particular, we then obtain the formula $b_{(1)} \cdot S_B(b_{(2)}) = \varepsilon_B(b)1_D$. Then since $(d \cdot b_{(1)} \cdot S_B(b_{(2)})) \cdot b_{(3)} = d \cdot b$, we also obtain that $(b' \cdot S_B(b_{(1)})) \cdot b_{(2)} = \varepsilon_B(b)b'$ for all $b, b' \in B$. (We remark that at this point, the proof of 1.2.5 would be completed.)

Choose a fixed $b \in B$ with $\varepsilon_B(b) = 1$, and write $\Delta_B(b) = \sum_i b_i \otimes b_i'$. Then $\{(b_i', S_B(b_i))\}$ gives us a finite projective basis of B, i.e. $b = \sum_i (b \cdot S_B(b_i)) \cdot b_i'$ for all $b \in B$. Hence ${}_DB$ is projective and finitely generated. On the other hand, ${}_DB$ is a generating module since $\sum_i b_i \cdot S_B(b_i') = 1_D$. So ${}_DB$ is a left Morita D-module, and defining A as $\mathrm{End}_D({}_DB)^{\mathrm{op}}$, we get that B is a D-A-equivalence bimodule.

Now we could proceed as in the proof of '3. implies 2.' for Proposition 1.2.17 to construct a comultiplication on C, but we will proceed by a different route. Namely, we first remark that $S_B(db) = S_B(b)S_D(d)$ (using again multiplications inside the associated linking algebra). Indeed: by the modularity of ε_B with respect to ε_D, we have

$$\begin{aligned} d'd_{(1)}b_{(1)}S_B(d_{(2)}b_{(2)}) &= d'\varepsilon_B(db) \\ &= d'\varepsilon_D(d)\varepsilon_B(b) \\ &= d'd_{(1)}\varepsilon_B(b)S_D(d_{(2)}) \\ &= d'd_{(1)}b_{(1)}S_B(b_{(2)})S_D(d_{(2)}), \end{aligned}$$

from which $S_B(db) = S_B(b)S_D(d)$ for all $b \in B$ and $d \in D$ easily follows. Then, since for any $c \in C$ and $b \in B$ with $\varepsilon_B(b) = 1$, we have $c = S_B(b_{(1)}) \cdot (b_{(2)}c)$, and since S_D is surjective on D, we have that S_B is in fact surjective onto C. So we can define a comultiplication on C by the formula

$$\Delta_C := (S_B \otimes S_B) \circ \Delta_B^{\mathrm{op}} \circ S_B^{-1},$$

and it is further immediate that $\Delta_C(cd) = \Delta_C(c)\Delta_D(d)$ for $c \in C$ and $d \in D$, using that S_D flips the comultiplication on D.

Since $A \cong C \underset{D}{\odot} B$, and $A \odot A \cong (C \odot C) \underset{D \odot D}{\odot} (B \odot B)$ by the remarks made in Proposition 1.2.17 and the third item of Corollary **??**, we can define a comultiplication Δ_A on A by putting

$$\Delta_A(c \cdot b) := (c_{(1)} \cdot b_{(1)}) \otimes (c_{(2)} \cdot b_{(2)}).$$

Alternatively, Δ_A can be defined by the defining property that

$$(d \otimes 1)\Delta_B(b)\Delta_A(a) := (d \otimes 1)\Delta_B(ba).$$

In any case, it is clear then that Δ_A will be a coassociative unital homomorphism, and that $\Delta_C(c)\Delta_B(b) = \Delta_A(cb)$ by definition.

We now show that A has a counit and bijective antipode.

Define $\varepsilon_C := \varepsilon_B \circ S_B^{-1}$. Then ε_C will be a counit for (C, Δ_C), and moreover $\varepsilon_C(cd) = \varepsilon_C(c)\varepsilon_D(d)$. Together with the corresponding identity for ε_B and the fact that we can identify A with $C \underset{D}{\odot} B$ as an A-A-bimodule, we can define a map $\varepsilon_A : A \to k$, uniquely determined by the property that $\varepsilon_A(cb) = \varepsilon_C(c)\varepsilon_B(b)$. Moreover, since $\varepsilon_D(bc) = \varepsilon_B(b)\varepsilon_C(c)$ by a similar argument as the ones already used, we get that ε_A is a homomorphism. It is also a counit for Δ_A: since $\varepsilon_B(ba) = \varepsilon_B(b)\varepsilon_A(a)$ by definition, we have

$$\begin{aligned} b(\iota_A \otimes \varepsilon_A)(\Delta_A(a)) &= (\iota_B \otimes \varepsilon_B)(\Delta_B(ba)) \\ &= ba, \end{aligned}$$

so $(\iota_A \otimes \varepsilon_A)\Delta_A(a) = a$. Similarly, $(\varepsilon_A \otimes \iota_A)\Delta_A(a) = a$.

Applying our discussion up to now to $(B, \Delta_B^{\mathrm{cop}})$, which is a comonoidal left D^{cop}-Morita module, we find a bijection $B \to C$, whose *inverse* we will denote by S_C, such that $b_{(2)} \cdot S_C^{-1}(b_{(1)}) = \varepsilon_B(b)1_D$ and $S_C^{-1}(b_{(2)}) \cdot b_{(1)} = \varepsilon_B(b)1_A$. We remark that the comultiplications on C defined by these antipodes agree, since we have the alternative expression $(d \otimes 1)\Delta_B(b)\Delta_C(c) = (d \otimes 1)\Delta_D(bc)$ for the comultiplication on C. Since $S_C(cd) = S_D(d)S_C(c)$ and $S_B(db) = S_B(b)S_D(d)$, we can define, again using that $C \underset{D}{\odot} B \cong A$,

$$S_A : A \to A : cb \to S_B(b)S_C(c),$$

which is clearly a bijective map. We show that S_A satisfies the antipode identity:

$$\begin{aligned} (cb)_{(1)}S_A((cb)_{(2)}) &= c_{(1)}b_{(1)}S_A(c_{(2)}b_{(2)}) \\ &= c_{(1)}b_{(1)}S_B(b_{(2)})S_C(c_{(2)}) \\ &= \varepsilon_B(b)c_{(1)}S_C(c_{(2)}) \\ &= \varepsilon_B(b)\varepsilon_C(c)1_A \\ &= \varepsilon_A(cb)1_A. \end{aligned}$$

Similarly, $S_A((cb)_{(1)})(cb)_{(2)} = \varepsilon_A(cb)1_A$, and we are done.

$\square$

By the previous proposition, it is also easy to see that if A and D are Hopf algebras, then an isomorphism class of comonoidal equivalence bimodules between A and D is completely determined by the isomorphism class of the associated right comonoidal Morita A-module, up to a group-like (hence necessarily invertible) element in D (compare Lemma 3.11 of [71]).

1.3 Monoidal co-Morita equivalence of Hopf algebras

As said at the beginning of this chapter, the theory of 'monoidal co-Morita equivalences' is, formally, completely dual to the theory that we have developed in the previous section. Therefore, we will not give complete proofs for the statements in this section (moreover, most of them are in the literature), although the statements can not be *deduced* from those in the previous section (except in the finite-dimensional case).

Definition 1.3.1. *Let A be a counital coalgebra. A* left counital comodule *(V, γ_V) for A consists of a k-vector space V and a linear map $\gamma_V : V \to A \odot V$, such that*

$$1. \quad (\iota_A \otimes \gamma_V)\gamma_V = (\Delta_A \otimes \iota_V)\gamma_V,$$

$$2. \quad (\varepsilon_A \otimes \iota_V)\gamma_V = \iota_V.$$

We will also write just the symbol for the underlying vector space V for a comodule, and γ_V for the associated comodule map, or vice versa, write γ for a left counital comodule, and then write the associated vector space by V_γ. We have the Sweedler notation for left counital comodules:

$$\gamma(v) = v_{(-1)} \otimes v_{(0)},$$
$$(\Delta_A \otimes \iota_V)(\gamma(v)) = v_{(-2)} \otimes v_{(-1)} \otimes v_{(0)}.$$

There is of course also the notion of a *right* counital comodule, which is then a couple (V, α_V) with $\alpha_V : V \to V \odot A$, satisfying the obvious identities. If α is a right counital comodule, we write

$$\alpha(v) = v_{(0)} \otimes v_{(1)}.$$

We can put a category structure on the collection of all left counital comodules: if V and W are two left counital comodules, we define

$$\mathrm{Mor}(V, W) = \{x : V \to W \mid (1 \otimes x)\gamma_V(v) = \gamma_W(xv)\}.$$

In case A is in fact a Hopf algebra, we also have a monoidal structure: we then define

$$\gamma_V \cdot \gamma_W : V \odot W \to A \odot (V \odot W) : v \otimes w \to (v_{(-1)} \cdot w_{(-1)}) \otimes v_{(0)} \otimes w_{(0)}.$$

Together with the trivial comodule $\eta_A : k \to A \odot k$, we get a strict monoidal category $(\text{A}-\text{CoMod}, \cdot, \eta_A)$ (when choosing the appropriate tensor product).

Again, we also have a strict monoidal category of right comodules (CoMod-A, $\cdot$, η_A).

Definition 1.3.2. *Let A and D be two counital coalgebras. We call them* co-Morita equivalent *(or Morita-Takeuchi equivalent) if their associated categories of right counital comodules are (k-linearly) equivalent, and call a particular such equivalence a co-Morita equivalence.*

When A and D are two Hopf algebras, we call them monoidally co-Morita equivalent *(or monoidally Morita-Takeuchi equivalent) if their associated monoidal categories of right counital comodules are monoidally equivalent. We call a particular such monoidal equivalence a* monoidal co-Morita equivalence *between A and D.*

Remark: One can show that in both situations, one obtains precisely the same notion if one only works with (CoMod^{fd}-A, $\cdot$, η_A), the full (monoidal) sub-category of finite-dimensional comodules.

Now we want to reformulate this notion again without using category theory.

Definition 1.3.3. *Let A be a Hopf algebra, and (B, α_B) a couple consisting of a unital algebra B with a counital right A-comodule structure α_B,* such that *α_B is a unital homomorphism. Then we call α_B a right* coaction *of A on B.*

The following concepts, studied in detail by Schauenburg in [71], are dual to the notion of 'comonoidal Morita module' and 'monoidal equivalence bimodule'.

Definition 1.3.4. *Let α_B be a right coaction of a Hopf algebra A on a unital algebra B. We call B a right* Galois object *if*

$$T_{1,\alpha_B} : B \odot B \to B \odot A : b \otimes b' \to (b \otimes 1)\alpha_B(b'),$$

called a Galois map *for α_B, is a bijection.*

Remark: Note that the bijectivity of T_{1,α_B} is equivalent to the bijectivity of the other Galois map

$$T_{\alpha_B,1} : B \odot B \to B \odot A : b \otimes b' \to \alpha_B(b)(b' \otimes 1),$$

by an easy argument. There is then no trouble or ambiguity in defining a *left* Galois object.

Definition 1.3.5. *Let A and D be Hopf algebras, B a unital algebra, α_B a right coaction of A on B, and γ_B a left coaction of D on B. We call (B, γ_B, α_B) a* bi-Galois object *if (B, γ_B) is a left, (B, α_B) a right Galois object, and α_B and γ_B commute, i.e.*

$$(\iota_D \otimes \alpha_B)\gamma_B = (\gamma_B \otimes \iota_A)\alpha_B.$$

Note that, in analogy with the previous section, we could also call a Galois object a '*monoidal co-Morita comodule*', and a bi-Galois object a '*monoidal equivalence bicomodule*'.

The following is a main result of [71] (which holds also when the antipode is not bijective, or, with some minor extra assumptions, when k is a general unital ring).

Proposition 1.3.6. *Let A and D be two Hopf algebras. Then there is a one-to-one correspondence between isomorphism classes of*

1. *monoidal co-Morita equivalences between A and D, and*
2. *bi-Galois objects between A and D.*

In particular, A and D are monoidally co-Morita equivalent iff there exists a bi-Galois object between them.

There is also a notion dual to that of a linking weak Hopf algebra.

Definition 1.3.7. *Let $(E, \{p_{ij}\})$ be a couple consisting of a weak Hopf algebra, together with a central decomposition $\{p_{ij}\}$ of the unit 1_E into four* non-trivial *constituents (so* $p_{11}, p_{12}, p_{21}, p_{22}$ *are four non-zero central elements,* $p_{ij} \cdot p_{kl} = \delta_{ik}\delta_{jl}\, p_{ij}$*, and* $\sum_{i,j} p_{ij} = 1$*). We call* $(E, \{p_{ij}\})$ *a* co-linking weak Hopf algebra *if*

$$\Delta_E(p_{ik}) = \sum_j p_{ij} \otimes p_{jk}.$$

When working with a co-linking weak Hopf algebra, we will again personalise its constituents, writing $p_{ij}E = E_{ij}$ or also $E_{11} = D$, $E_{21} = C$, $E_{12} = B$ and $E_{22} = A$. We also write

$$\Delta_{ij}^k : E_{ij} \to E_{ik} \odot E_{kj} : x_{ij} \to (p_{ik} \otimes p_{kj})\Delta_E(x_{ij}),$$

which we personalise as $\Delta_{11}^1 = \Delta_D$, $\Delta_{12}^1 = \gamma_B$, $\Delta_{21}^1 = \gamma_C$, $\Delta_{12}^2 = \alpha_B$, $\Delta_{21}^2 = \alpha_C$, $\Delta_{11}^2 = \beta_D$, $\Delta_{22}^1 = \beta_A$, and finally $\Delta_{22}^2 = \Delta_A$. We call β_A and β_D the *external comultiplication* maps.

It is not hard to show, using the general theory of weak Hopf algebras, that the counit of E vanishes on B and C. Denoting by ε_A and ε_D the restriction of ε_E to A, resp. D, it is also easy to see that they provide counits on the coalgebras A and D, and that A and D then become bialgebras. Further, S_E will restrict to maps $S_A : A \to A$, $S_B : B \to C$, $S_C : C \to B$ and $S_D : D \to D$, with S_A and S_D then providing antipodes for the bialgebras A and D. Hence A and D are in fact Hopf algebras.

Definition 1.3.8. *Let A and D be two Hopf algebras. We call a quadruple* $(E, \{p_{ij}\}, \Phi_A, \Phi_D)$ *a* co-linking weak Hopf algebra between A and D *if* $(E, \{p_{ij}\})$ *is a co-linking weak Hopf algebra, and if the* Φ_A *and* Φ_D *are comultiplication preserving isomorphisms* $A \underset{\Phi_A}{\cong} (E_{22}, \Delta_{22}^2)$ *and* $D \underset{\Phi_D}{\cong} (E_{11}, \Delta_{11}^1)$.

Again, we mostly suppress the notation for the isomorphisms Φ_A and Φ_D in the previous definition.

As for linking weak Hopf algebras, one can give a quasi-classical interpretation of co-linking weak Hopf algebras, by considering them as non-commutative *function* spaces on a quantum groupoid with a classical object space consisting of two objects. The composition of arrows is now encoded in the comultiplication of the co-linking weak Hopf algebra.

The following shows that there is no real difference between bi-Galois objects and co-linking weak Hopf algebras between.

Proposition 1.3.9. *Let A and D be two Hopf algebras. Then there is a one-to-one correspondence between isomorphism classes of*

1. *co-linking weak Hopf algebras between A and D, and*
2. *bi-Galois objects between A and D.*

Proof. Let $(E, \{p_{ij}\})$ be a co-linking weak Hopf algebra between A and D. Then from the coassociativity of Δ_E and its behavior with respect to the p_{ij}, it follows that $(\iota_B \otimes \Delta_A)\alpha_B = (\alpha_B \otimes \iota_A)\alpha_B$. Since ε_A is the restriction of ε_E to A, it is also easy to see that α_B is counital, hence provides a right coaction of A on B. Similarly, γ_B is a left coaction of D on B. The coassociativity of Δ_E also shows immediately that γ_B and α_B commute.

Now write $\beta_A(a) = a_{[1]} \otimes a_{[2]}$. Then the antipode identity on E gives that

$$b' b_{(0)} S_C(b_{(1)[1]}) \otimes b_{(1)[2]} = b' \otimes b,$$

$$b S_C(a_{[1]}) a_{[2](0)} \otimes a_{[2](1)} = b \otimes a,$$

which lets us conclude that

$$T_{1,\alpha_B} : B \odot B \to B \odot A : b \otimes b' \to (b \otimes 1)\alpha_B(b')$$

is a bijection. Similarly, one shows that

$$T_{\gamma_B,2} : B \odot B \to D \odot B : b \otimes b' \to \gamma_B(b)(1 \otimes b')$$

is a bijection. Hence (B, γ_B, α_B) is a bi-Galois object.

Now let (B, γ_B, α_B) be a bi-Galois object. Denote

$$E = D \oplus C \oplus B \oplus A,$$

where $C = B^{\mathrm{op}}$. We give E the direct sum algebra structure. We denote the units of D, C, B and A, seen as central orthogonal idempotents in E, respectively by p_{11}, p_{21}, p_{12} and p_{22}. We will construct a comultiplication Δ_E on E. Denote

$$\tilde{\beta}_A(a) := T^{-1}_{1,\alpha_B}(1 \otimes a),$$

and write $\tilde{\beta}_A(a) = a^{[1]} \otimes a^{[2]}$. Further write

$$\beta_A : A \to C \odot B : a \to (a^{[1]})^{\mathrm{op}} \otimes a^{[2]},$$

which we then write as $\beta_A(a) = a_{[1]} \otimes a_{[2]}$. In a similar way, one constructs a map

$$\beta_D : D \to B \odot C.$$

As in [79], one can check that β_A and β_D are unital homomorphisms. Further write

$$\alpha_C : C \to A \odot C : b^{\mathrm{op}} \to S_A^{-1}(b_{(1)}) \otimes b_{(0)}^{\mathrm{op}},$$

$$\gamma_C : C \to C \odot D : b^{\mathrm{op}} \to b_{(0)}^{\mathrm{op}} \otimes S_D^{-1}(b_{(-1)}),$$

then α_C is a left coaction by A and γ_C a right coaction by D. We then define a map

$$\Delta_E : E \to E \odot E,$$

which is given on the different components of E as

$$\Delta_E(a) = \beta_A(a) + \Delta_A(a),$$

$$\Delta_E(b) = \gamma_B(b) + \alpha_B(b),$$

$$\Delta_E(c) = \alpha_C(c) + \gamma_C(c),$$

$$\Delta_E(d) = \Delta_D(d) + \beta_D(d).$$

It is then immediate that Δ_E is a (non-unital) homomorphism, and that the unit is weakly comultiplicative. One can also show that Δ_E is coassociative, although we refrain from carrying out this computation in full here (one can prove this piecewise, using coassociativity, the coaction property, the commuting property between α_B and α_C, and formulas as in Lemma 2.1.7 in [76]).

Now define $\varepsilon_E(d \oplus c \oplus b \oplus a) = \varepsilon_D(d) + \varepsilon_A(a)$. Then it is trivial to see that ε_E is a counit for the coalgebra E, and that it is moreover weakly multiplicative. Hence E is in fact a monoidal and comonoidal weak bialgebra.

Now we use a result from [77], which is an analogue of Proposition 1.2.5 for weak bialgebras: if E is a monoidal and comonoidal weak bialgebra, then it is a weak Hopf algebra (possibly with non-bijective antipode) iff

$$E \underset{E^s}{\odot} E \to \Delta_E(1_E)(E \odot E) : x \underset{E^s}{\otimes} y \to \Delta_E(x)(1_E \otimes y)$$

is an isomorphism. (Alternatively, we could have established this proposition specifically for our situation, with the same techniques already used extensively in the previous section.) Applied to the weak bialgebra already obtained, this map splits into 8 maps

$$E_{ik} \odot E_{jk} \to E_{ij} \odot E_{jk} :$$

$$x_{ij} \odot y_{jk} \to \Delta^{j}_{ik}(x_{ij})(1_{ij} \otimes y_{jk}),$$

all of which should be isomorphisms. But 4 of them (for example, the ones with $k = 1$) can be omitted by symmetry reasonings, and then 2 further ones can be omitted by 1.2.5 and by the fact that α_B is a Galois object. Thus we only have to check if

$$B \odot A \to B \odot A : b \otimes a \to \alpha_B(b)(1 \otimes a)$$

is bijective, but this is true for *any* coaction, as is easily verified, and if

$$A \odot B \to C \odot B : a \otimes b \to \beta_A(a)(1 \otimes b)$$

is bijective. But using again the formulas of Lemma 2.1.7 in [76], we find that

$$C \odot B \to A \odot B : c \otimes b \to (\iota_A \otimes S_C)(\gamma_C(c))(1 \otimes b)$$

is an inverse for this last map. The same reasoning, applied to E^{cop}, shows that the associated antipode S_E is in fact bijective.

Before proving that the two operations introduced so far are inverses of each other, we first give some further comments about this antipode S_E. It is for example easy to see, using once more the formulas of Lemma 2.1.7 in [76], that S_C, which is S_E restricted to C, is simply $C \to B : b^{\text{op}} \to b$. However, the restriction S_B of S_E to B will be of the form $b \to \theta_B(b)^{\text{op}}$, where θ_B is a certain automorphism of the algebra B. This automorphism θ_B was called the *Grunspan map* in Definition 3.5 of [75], and was actually defined for quantum torsors (which are in one-to-one correspondence with our co-linking weak Hopf algebras, but which have a slimmer axiom system). See also the original paper [43]. We will see this automorphism appear again in the third chapter, but we call it there the antipode squared associated to a Galois object (for obvious reasons). Our definition of it will however be given in a different way than in [75].

Now, to show that these two operations we introduced are inverses of each other, we only have to show that a co-linking weak Hopf algebra $(E, \{p_{ij}\})$ between A and D is completely determined by its associated bi-Galois object

(B, γ_B, α_B). But $S_C : C \to B$ provides an anti-isomorphism between C and B, hence the algebra structure on C is completely determined. Since

$$\Delta_E \circ S_E = (S_E \otimes S_E)\Delta_E^{\mathrm{op}},$$

the comultiplication on the C-part is completely determined. Finally, since

$$A \odot B \to C \odot B : a \otimes b \to \beta_A(a)(1 \otimes b)$$

has

$$C \odot B \to A \odot B : c \otimes b \to (\iota_A \otimes S_C)(\gamma_C(c))(1 \otimes b)$$

as its inverse, β_A, and by symmetry, β_D are completely determined. Hence all structure involving C is fixed, and we are done.

□

The following result, again due to Schauenburg ([71]), is the dual of Proposition 1.2.18.

Proposition 1.3.10. *Let A be a Hopf algebra, and let B be a right Galois object for A. Then there exists a Hopf algebra D which completes it to a bi-Galois object between A and D. It is unique in the following sense: if D_1 is another Hopf algebra, and B is also a bi-Galois object between A and D, then there exists an isomorphism $\Phi_D : D \to D_1$ of Hopf algebras, such that*

$$\gamma_{D_1} = (\Phi_D \otimes \iota)\gamma_D.$$

We now introduce the following generalization of the concept of a Galois object.

Definition 1.3.11. *Let A be a Hopf algebra, α_B a right coaction of A on a unital algebra B. The* algebra of coinvariants *for α_B is the set of elements $b \in B$ for which $\alpha_B(b) = b \otimes 1$ (which are then called* coinvariants*).*

It is easily seen that the set of coinvariants is really a unital subalgebra of B.

Definition 1.3.12. *Let A be a Hopf algebra, α_B a right coaction of A on a unital algebra B. We call α_B a* Galois coaction *if, denoting by F the algebra of coinvariants, a (generalized) Galois map*

$$B \underset{F}{\odot} B \to B \odot A : b \underset{F}{\otimes} b' \to (b \otimes 1)\alpha_B(b')$$

is bijective.

As for Galois objects, one can as well put the other Galois map

$$B \underset{F}{\odot} B \to B \odot A : b \underset{F}{\otimes} b' \to \alpha_B(b)(b' \otimes 1)$$

in the definition.

We note that many results concerning Galois objects also hold for Galois coactions, with the main difference that the rôle of D is now played by a 'Hopf algebroid' (see also the remark following Definition 1.2.16).

We end with the following minor observation. First remark that one can also define right coactions for weak Hopf algebras. One such definition goes as follows. We can coact on the right on unital algebras B equipped with a unital anti-homomorphism s_B of L, the algebra of objects of the weak Hopf algebra A, into B. Then such an algebra becomes an L-bimodule, by composing s_B with either left or right multiplication. Consider also A as an L-bimodule by composing t with multiplication to the left or right. Note then that $(B \underset{L}{\odot} A)^L$, the set of L-central elements in $(B \underset{L}{\odot} A)$, becomes an algebra under the obvious (factorwise) multiplication rule. Then a coaction of A on B consists, apart from s_B, of a unital homomorphism $\alpha_B : B \to (B \underset{L}{\odot} A)^L$, satisfying $\alpha_B(s_B(x)) = 1_B \underset{L}{\otimes} s_A(x)$, for $x \in L$, and the natural coassociativity relation, which, in Sweedler notation, reads

$$b_{(0)(0)} \underset{L}{\otimes} b_{(0)(1)} \underset{L}{\otimes} b_{(1)} = b_{(0)} \underset{L}{\otimes} b_{(1)(1)} \underset{L}{\otimes} b_{(1)(2)}$$

for all $b \in B$ (where we have also interpreted Δ_A as a map $A \to (A \underset{L}{\odot} A)^L$, with L-bimodule structure as on B but using now the map s_A, and where we have also identified $(B \underset{L}{\odot} A) \underset{L}{\odot} A$ with $B \underset{L}{\odot} (A \underset{L}{\odot} A)$, although this actually requires some care), together with the counital assumption $x_{(0)} \underset{L}{\otimes} \mathcal{E}_t(x_{(1)}) = x \underset{L}{\otimes} 1_A$ for $x \in B$. Note that by the last property, α_B is automatically faithful. For such a coaction, one can again define the algebra F of coinvariants as the set of those elements $b \in B$ for which $\alpha_B(b) = b \underset{L}{\otimes} 1_A$. Then one can form a Galois map

$$B \underset{F}{\odot} B \to B \underset{L}{\odot} A : b \underset{F}{\otimes} b' \to bb'_{(0)} \underset{L}{\otimes} b'_{(1)}.$$

We call the coaction Galois, when s_B is faithful and this Galois map is an isomorphism.

Now take $A = M_2(k)$, endowed with the weak Hopf algebra structure of the groupoid algebra of the connected groupoid $\mathbb{2}$ with two points and four arrows, that is, with its usual product structure and with the 'trivial' comultiplication $\Delta_A(e_{ij}) = e_{ij} \otimes e_{ij}$. Then we note that L is k^2, embedded as the commutative subalgebra of diagonal elements in $M_2(k)$, and that $s = t$. Now let E be a unital algebra equipped with a coaction by $M_2(k)$. Then $e = s_E(e_2)$ provides an idempotent in E. Let $E = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be the associated decomposition of E with respect to e. Note that if s_E is faithful, then e is not trivial, so neither A or D are the zero algebra. Further, since α_E has range in $(E \underset{k^2}{\odot} M_2(k))^{k^2}$, we see that for $x_{ij} \in E_{ij}$, we have $\alpha_E(x_{ij}) = x'_{ij} \underset{k^2}{\otimes} e_{ij}$ for some $x'_{ij} \in E_{ij}$ (being careful to use the right module structures!). But since α_E is a coaction, and α_E is faithful, it is easy to see that in fact $x'_{ij} = x_{ij}$. Hence a coaction by $M_2(k)$ is completely determined by the idempotent e inside E.

Let us now consider however what happens when the coaction is Galois. Remark first that the algebra of coinvariants can easily be verified to be the algebra $D \oplus A$ inside E. Then $E \underset{D\oplus A}{\odot} E$ can naturally be identified with

$$\begin{pmatrix} (E_{11} \underset{D}{\odot} E_{11}) & (E_{11} \underset{D}{\odot} E_{12}) \\ (E_{21} \underset{D}{\odot} E_{11}) & (E_{21} \underset{D}{\odot} E_{12}) \end{pmatrix} \oplus \begin{pmatrix} (E_{12} \underset{A}{\odot} E_{21}) & (E_{12} \underset{A}{\odot} E_{22}) \\ (E_{22} \underset{A}{\odot} E_{21}) & (E_{22} \underset{A}{\odot} E_{22}) \end{pmatrix}.$$

On the other hand, $E \underset{k^2}{\odot} M_2(k)$ is easily seen to coincide with $E \oplus E$, sending $x_{ij} \underset{k^2}{\otimes} e_{ik}$ to x_{ij} in the k-th component. The Galois map will then coincide exactly with the multiplication map of E, restricted to each summand. From this, we conclude that α_E will be a Galois coaction iff E is a unital linking algebra. Another way of saying this is that unital linking algebras are precisely strong $\mathbb{2}$-graded unital algebras.

It is then further easily noted that linking weak Hopf algebras are exactly those weak linking Hopf algebras equipped with a Galois coaction of $M_2(k)$, in such a way that the coaction and the comultiplication commute. One can even better appreciate the situation in the case of co-linking weak Hopf algebras. Now we should look at the weak Hopf algebra k^4, which is the *function* algebra of the groupoid $\mathbb{2}$, where the comultiplication on the Dirac function δ_{ij} (corresponding to the arrow from i to j) is given as $(\delta_{i1} \otimes \delta_{1j}) + (\delta_{i2} \otimes \delta_{2j})$. Then co-linking weak Hopf algebras E are precisely those weak

Hopf algebras containing k^4 as a sub-weak Hopf algebra. This is of course what one should expect for the dual situation (compare Proposition 7.1.4).

1.4 Special cases and examples

As mentioned already, in the finite-dimensional case there is a very easy direct correspondence between Galois objects (monoidal co-Morita modules) and Galois coobjects (comonoidal Morita modules): one simply has to consider the vector space dual and transpose all structure. In more detail: let A be a finite-dimensional Hopf algebra. Then its k-linear dual A^* obtains a Hopf algebra structure, by defining

$$(M_{A^*}(\omega_1 \otimes \omega_2))(a) = (\omega_1 \otimes \omega_2)(\Delta_A(a))$$

and

$$\Delta_{A^*}(\omega)(a \otimes a') = \omega(a \cdot a'),$$

where $\omega_1, \omega_2, \omega \in A^*$, and where we have identified $(A \odot A)^*$ with $A^* \odot A^*$. Then ε_A provides the unit of A^*, evaluation in 1_A the counit, and the transpose of the antipode S_A the antipode S_{A^*}. We denote this Hopf algebra by $\widehat{A}$, and call it the Hopf algebra dual to A.

Now if (B, α_B) is a right A-Galois object, we can transpose α_B to obtain a right $\widehat{A}$-module structure on B^*, and we can transpose the multiplication on B to obtain a comultiplication Δ_{B^*} on B^*. It is then trivial to check that B^* is in fact a right $\widehat{A}$-module coalgebra, and even a right comonoidal Morita module. We then denote it by $\widehat{B}$. Similarly, starting from a right comonoidal Morita module, one produces a Galois object for the dual Hopf algebra by considering the dual space.

However, in the finite-dimensional case, comonoidal Morita are in fact quite trivial as a right A-module: they are simply a copy of A with its right module structure by right multiplication. We put this in the form of a definition.

Definition 1.4.1. *Let A be a Hopf algebra. A comonoidal right Morita A-module B is called* cleft *when $B_A \cong A_A$.*

If B is a cleft comonoidal right Morita A-module, we can put $\Omega = \Delta_B(1_A)$, where we have simply identified B_A with A_A for convenience. By the bijectivity of the Galois map, Ω will be an invertible element of $A \odot A$. Since Δ_B is coassociative, Ω will satisfy the 2-cocycle identity:

$$(\Omega \otimes 1_A)((\Delta_A \otimes \iota_A)(\Omega)) = (1_A \otimes \Omega)((\iota_A \otimes \Delta_A)(\Omega)).$$

Conversely, any 2-cocycle, i.e. any invertible element Ω in $A \odot A$ satisfying this equality[4] is easily seen to give rise to a (cleft) comonoidal right Morita A-module structure on A itself, by putting

$$\Delta_B(a) := \Omega\Delta_A(a).$$

Two cleft comonoidal right Morita A-modules with associated 2-cocycles Ω_1 and Ω_2 will then be isomorphic precisely when they are cohomologous, that is, when there exists an invertible element $u \in A$ such that $\Omega_2\Delta_A(u) = (u \otimes u)\Omega_1$. Finally, it is also easy to see how the reflected Hopf algebra along a cleft comonoidal right Morita A-module with associated 2-cocycle Ω looks like: this is simply the algebra A with the new coproduct

$$\Delta_D(a) = \Omega\Delta_A(a)\Omega^{-1}.$$

Dually, one also has the notion of cleft Galois objects: while we do not give the precise definition, we mention that they can again be characterized in terms of '2-cocycles', which are now however functions $\omega : A \odot A \to k$, satisfying among other conditions a natural 2-cocycle identity. It is easily guessed that in case A is finite-dimensional, then ω, interpreted as an element of $\widehat{A} \odot \widehat{A}$, will be a 2-cocycle for $\widehat{A}$ (as in the previous paragraph).

Apart from this quite general type of example, we can of course not neglect to mention the Galois objects which inspired this name-giving, namely the (finite) Galois field extensions. First note that associated to any finite group $\mathfrak{G}$, there corresponds a Hopf algebra in the following way. The associated algebra is given by the set of all functions from $\mathfrak{G}$ to k, with pointwise addition, multiplication and scalar multiplication. Note then that $k(\mathfrak{G}) \underset{k}{\odot} k(\mathfrak{G}) \cong k(\mathfrak{G} \times \mathfrak{G})$, in a natural way. Thus we can define a comultiplication on $k(\mathfrak{G})$ by saying that $\Delta_{k(\mathfrak{G})}(f)$, for $f \in k(\mathfrak{G})$, should be the function on $\mathfrak{G} \times \mathfrak{G}$ which sends (g, h) to $f(gh)$. Then the associativity of $\mathfrak{G}$ shows that $\Delta_{k(\mathfrak{G})}$ is coassociative. The counit is given by evaluation in the unit element of $\mathfrak{G}$, while the antipode is given by $(S_{k(\mathfrak{G})}(f))(g) = f(g^{-1})$. Then actions of the group $\mathfrak{G}$ on an algebra are in one-to-one correspondence with coactions of the Hopf algebra $k(\mathfrak{G})$, by letting an action α of $\mathfrak{G}$ on an algebra B correspond to the coaction $B \to B \odot k(\mathfrak{G})$ which sends $b \in B$ to $\sum_{g\in\mathfrak{G}} \alpha_g(b) \otimes \delta_g$, where δ_g is the Dirac function in the point $g \in \mathfrak{G}$.

[4]One often also asks the normalization condition $(\varepsilon_A \otimes \iota_A)\Omega = (\iota_A \otimes \varepsilon_A)(\Omega) = 1_A$, so to have $\varepsilon_B = \varepsilon_A$. However, this is merely a matter of convenience, since any 2-cocycle can be perturbed to a normalized one.

Now suppose $k \subseteq K$ is a finite field extension. Then it is an easy exercise, using basic Galois theory, to prove that this extension is Galois, i.e., k is the fixed point algebra of all automorphisms of K leaving k element-wise fixed, if and only if the associated coaction by the function algebra of the automorphism group of $k \subseteq K$ on K, considered as a k-algebra, is Galois.

However, it should be mentioned that it is quite possible that a finite dimensional Hopf algebra over a field k acts by a Galois coaction α_K on a field $K \supseteq k$, without this field extension being Galois! Such Hopf algebras will then necessarily be commutative, but can of course not be of the form $k(G)$ for some group G (although they *do* become of this form when one amplifies them with a big enough field extension of k). We refer to the article [42] for more information.

Next, we address the natural question whether there exist non-cleft Galois objects. This was in fact not clear at all from the beginning. The existence of these was established in [14], but the first concrete examples were obtained by Bichon in [9]. Then in [10], examples were also found in the C*-algebra setting, where there are even extra conditions on both the Hopf algebra and the algebra acted upon, requiring them for example to have a well-behaving *-structure, and a coaction respecting this *-structure. We will treat some of the *-theory in the third chapter.

Finally, we end with the following remark, already alluded to in the introduction. Let k be an algebraically closed field of characteristic 0, for example $\mathbb{C}$. Then any finite dimensional Hopf algebra A which is cocommutative, i.e., which satisfies $\Delta_A = \Delta_A^{\mathrm{op}}$, is of the form kG for some finite group G. Here kG is the group algebra of G, endowed with the Hopf algebra structure for which $\Delta_{kG}(g) = g \otimes g$. Let $\Omega \in kG \odot kG$ be a 2-cocycle. Then in general, there is no reason to expect that the Ω-twisted Hopf algebra is again cocommutative, and the extra requirement necessary for cocommutativity to hold is easily derived: $(\Omega^{\mathrm{op}})^{-1}\Omega$, where Ω^{op} is Ω with its legs interchanged, should commute with all $g \otimes g$. Is it possible to find a non-trivial 2-cocycle (i.e., not cohomologous to the trivial 2-cocycle $1_{kG} \otimes 1_{kG}$) satisfying this condition? The surprising answer is that such groups and 2-cocycles do indeed exist. Even more: the reflected Hopf algebra, which is then of the form kH for some (unique) finite group H, does not have to be isomorphic to kG, i.e., H does not have to be isomorphic to G. In categorical terms, this means that H and G have almost completely identical representation categories, for the categories can not be distinguished as monoidal categories. However, they can be distinguished as *symmetric* monoidal categories. By a symmetric monoidal category, we mean a monoidal category $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}})$ together with an extra natural transformation σ from $\otimes_{\mathcal{C}}$ to $\otimes_{\mathcal{C}} \circ \Sigma_{\mathcal{C},\mathcal{C}}$, satisfying certain relations. In case of the representation category of a finite group, this is simply the natural transformation for which

$$\sigma_{V,W} : V \odot W \to W \odot V : \sum_i v_i \otimes w_i \to \sum_i w_i \otimes v_i.$$

Again, as already mentioned in the introduction, the representation category of a finite group, as a *symmetric* monoidal category, completely remembers the group, by an (easy) corollary to a theorem by Deligne (see [24], although *this* particular result was in fact known much earlier already). For examples of groups having monoidally equivalent representation categories, we refer to [38], where a whole family of them is obtained, and to [48], where examples are provided of finite groups having even the same monoidal C*-category of unitary representations.

Chapter 2

Preliminaries on algebraic quantum groups

In this chapter, we first discuss some notions concerning non-unital algebras, and explain how one can develop Morita theory for them. Then we introduce multiplier Hopf algebras ([92]), which are genuine generalizations of Hopf algebras to the setting of non-unital algebras. After this, we recall the main results concerning algebraic quantum groups ([93]), which are a nice behaving subclass of the class of multiplier Hopf algebras, allowing for example for a duality theory. We also spend some time on a result, obtained in [21], concerning the further structure of algebraic quantum groups in presence of a well-behaving *-structure. We end with stating the definition of a Galois coaction for an algebraic quantum group (which is taken from [97]).

Throughout this chapter, k is again an arbitrary fixed field, unless otherwise stated.

2.1 Non-unital algebras

Definition 2.1.1. *Let A be an algebra.*

- *We call A* firm *if*

$$A \underset{A}{\otimes} A \to A : a \underset{A}{\otimes} a' \to aa'$$

is a bijection.

- *We call A* left (resp. right) non-degenerate *when A has no non-zero left (resp. right) zero multipliers, i.e. elements $a \in A$ satisfying $aa' = 0$ for all $a' \in A$ (resp. $a'a = 0$ for all $a' \in A$). We call A non-degenerate when it is both left and right non-degenerate.*

- *We call A* idempotent *when $A \cdot A = A$.*

- *We say that A has* left (resp. right) local units*, when for each $a \in A$ there exists an element $a' \in A$ such that*

$$a' \cdot a = a \ (\textit{resp. } a \cdot a' = a).$$

  We say that A has local units *when it has left and right local units.*

- *We say that A has a* left (resp. right) unit *when there exists an element $1_A \in A$ such that*

$$1_A \cdot a = a \ (\textit{resp. } a \cdot 1_A = a) \qquad \textit{for all } a \in A.$$

From (for example) Lemma 2.2 of [101], it follows that if A is an algebra with left (resp. right) local units, then for each finite subset $\{a_i\}$ of A, one can find $a \in A$ such that $a \cdot a_i = a_i$ (resp. $a_i \cdot a = a_i$), and, from Corollary 2.5 of that paper, that if A is an algebra with left and right local units, then for each finite subset $\{a_i\}$ of A, one can find $a \in A$ such that $a \cdot a_i = a_i = a_i \cdot a$. More trivially, if A has both a left and a right unit, then A is unital.

It is further immediate that a firm algebra is idempotent, that an algebra which has left or right local units is firm, and that an algebra with local units is non-degenerate. The notions 'being non-degenerate and idempotent' and 'being firm' are instances of nice regularity conditions which can be put onto a non-unital algebra, but unfortunately, they are unrelated in general. We present some examples to illustrate this fact.

The first two examples show that they are unrelated even in the commutative setting, and that they really 'complement' each other.

Example 2.1.2. *Let A be the quotient of the semi-group algebra of $(\mathbb{Q}_0^+, +)$, dividing out the generators corresponding to rational numbers which are strictly greater than 1. Then A is a firm algebra which is degenerate.*

Proof. Denote the generators of A as a k-vector space as α^i, where $0 < i \leqslant 1$ is a rational number. Then the multiplication rule is given by k-linearly extending the defining relations $\alpha^i \cdot \alpha^j = a^{i+j}$ when $i + j \leqslant 1$, and $\alpha^i \cdot \alpha^j = 0$ when $i + j > 1$.

It is easy to check that $a = \alpha^1$ satisfies $a \cdot a' = 0$ and $a' \cdot a = 0$ for all $a' \in A$, so A is both left and right degenerate. Since $\alpha^i = (\alpha^{i/2})^2$ for any i, it is also clear that A is idempotent.

Now let I be a finite subset of $\mathbb{Q} \cap]0, 1]$, and suppose we have an $I \times I$-indexed family of elements k_{ij} in k which satisfy $\sum_{i,j} k_{ij} \alpha^i \cdot \alpha^j = 0$. Denote

$$i_0 = \frac{1}{2} \cdot \min(I \cup \{i + j - 1 \mid i, j \in I \text{ and } i + j > 1\}).$$

Then it is clear that $\sum_{i,j} k_{ij} \alpha^{i-i_0} \cdot \alpha^j$ is a well-defined sum in A, equal to 0. Hence

$$\begin{aligned} \sum_{i,j} k_{ij} \alpha^i \underset{A}{\otimes} \alpha^j &= \sum_{i,j} k_{ij} \alpha^{i_0} \cdot \alpha^{i-i_0} \underset{A}{\otimes} \alpha^j \\ &= \alpha^{i_0} \underset{A}{\otimes} (\sum_{i,j} k_{ij} \alpha^{i-i_0} \cdot \alpha^j) \\ &= 0, \end{aligned}$$

so A is firm.

□

Example 2.1.3. *Let A be the quotient of the semi-group algebra of $(\mathbb{Q}_0^+, +)$, dividing out the generators corresponding to rational numbers which are greater* or equal *to 1. Then A is a non-degenerate algebra which is not firm.*

Proof. Denote again the generators of A as a k-vector space as α^i, where now $0 < i < 1$ is a rational number.

It is again easy to see that A is idempotent. To see that it is non-degenerate: suppose that I is a finite subset of $\mathbb{Q} \cap]0, 1[$, that k_i is an I-valued family of elements of k, and that $a = \sum_i k_i \alpha^i$ satisfies $a \cdot a' = 0$ for all $a' \in A$. Then, since $\mathbb{Q}_0^+$ satisfies the cancelation law, this implies that for $i \in I$, either $k_i = 0$ or $i + j \geqslant 1$ for all $j \in \mathbb{Q}_0^+$. But the latter implies that already $i \geqslant 1$. Hence $a = 0$, and A is non-degenerate.

We show now that A is not firm, by proving that $\alpha^{1/2} \underset{A}{\otimes} \alpha^{1/2} \neq 0$, although $\alpha^{1/2} \cdot \alpha^{1/2} = 0$. Indeed: if $\alpha^{1/2} \underset{A}{\otimes} \alpha^{1/2}$ were zero, then we can find a finite subset I of $\mathbb{Q} \cap]0,1[$ and an $I \times I \times I$-valued family of elements k_{rst} of k, such that

$$\alpha^{1/2} \otimes \alpha^{1/2} = \sum_{r,s,t} k_{rst}(\alpha^r \alpha^s \otimes \alpha^t - \alpha^r \otimes \alpha^s \alpha^t).$$

It is easily seen that we can assume all non-zero k_{rst} to satisfy $r + s + t = 1$. Let i_0 be a strictly positive rational number strictly smaller than $1/2$ and strictly smaller than any element of I. Then also

$$\alpha^{1/2-i_0} \otimes \alpha^{1/2} = \sum_{r,s,t} k_{rst}(\alpha^{r-i_0} \alpha^s \otimes \alpha^t - \alpha^{r-i_0} \otimes \alpha^s \alpha^t).$$

Applying the multiplication operator M_A, this would lead us to $\alpha^{1-i_0} = 0$, a contradiction.

□

The following gives an example of what can go wrong in a purely non-commutative situation:

Example 2.1.4. *Let* $A = \begin{pmatrix} \mathbb{C} & \mathbb{C} \\ 0 & 0 \end{pmatrix}$ *as subalgebra of the* $\mathbb{C}$*-algebra* $M_2(\mathbb{C})$ *of 2 by 2 matrices over* $\mathbb{C}$*. Then* A *is firm but degenerate.*

Proof. The algebra A is firm, since $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is a left unit. But since A has the left zero multiplier $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, it is degenerate.

□

Remark: We borrow the terminology of a *firmness* from [15] (where, in the setting of *rings*, it is said to be due to Quillen). In [41], firm algebras are also called *regular algebras.*

In the literature, especially the notion of a firm algebra has been studied, for example in connection with Morita theory (cf. [41]). On the other hand, in the theory of multiplier Hopf algebras (cf. [92]), the notion of non-degeneracy is the main regularity condition. A priori, it is not clear what could be the nicest possible, yet general enough regularity condition on a non-unital algebra. But it turns out that the *algebras underlying multiplier Hopf algebras*, which are more or less the only algebras we will encounter further on, *have local units*, so that *a posteriori* we needn't really worry about such questions concerning regularity, since having local units is already a strong condition.

It is not difficult to check that the tensor product algebra of two algebras which satisfy one of the regularity conditions introduced (such as firmness, non-degeneracy, being idempotent, having local units) is of the same type (the proof to check preservation of firmness is the most involved, see [41]).

Definition 2.1.5. *A* *-algebra *consists of a* $\mathbb{C}$*-algebra* A*, equipped with an anti-multiplicative, anti-linear*[1] *involution*

$$^* : A \to A : a \to a^*.$$

It is called positive*, if* $a^*a = 0$ *implies* $a = 0$*. It is called* completely positive *if* $\sum_i a_i^* a_i = 0$ *implies* $a_i = 0$ *for all* i*.*

Note that a positive *-algebra is automatically non-degenerate.

When A and B are two *-algebras, also the tensor product algebra $A \odot B$ is a *-algebra, by defining $(a \otimes b)^* := a^* \otimes b^*$ and extending anti-linearly. Then, denoting with $M_n(\mathbb{C})$ the n-by-n-matrices over $\mathbb{C}$ with its canonical *-algebra structure, it is easy to see that A is a completely positive *-algebra iff $A \odot M_n(\mathbb{C})$ is positive for each $n \in \mathbb{N}$, whence the name. It is an open problem if the tensor product algebra of two non-degenerate *-algebras is again non-degenerate in the absence of sufficiently many hermitian positive functionals on the two *-algebras.

It is further clear what is meant by a *-homomorphism between *-algebras.

Definition 2.1.6. *Let* A *be an algebra. The* multiplier algebra $M(A)$ *of* A *is the unital subalgebra of* $End_k(A) \oplus End_k(A)^{op}$*, consisting of those* (l, r^{op}) *for which*

$$r(a) \cdot a' = a \cdot l(a'), \qquad \text{for all } a, a' \in A.$$

It is convenient to write an element (l, r^{op}) of $M(A)$ as m, and to write $l(a) = m \cdot a$ and $r(a) = a \cdot m$ for $a \in A$. Then we have a homomorphism

[1] an $\mathbb{R}$-linear map f between $\mathbb{C}$-vector spaces is called *anti-linear* when $f(cx) = \bar{c}f(x)$ for $c \in \mathbb{C}$.

of A into $M(A)$, by sending $a \in A$ to (l_a, r_a^{op}), where $l_a(a') = a \cdot a'$ and $r_a(a') = a' \cdot a$ for $a' \in A$. When A is non-degenerate, this homomorphism is faithful. In this case, we will identify A with its part inside $M(A)$, and we will then denote the unit of $M(A)$ also by 1_A (instead of $1_{M(A)}$).

When A is a *-algebra, then the multiplier algebra $M(A)$ for the underlying algebra has a natural *-structure, making the natural homomorphism $A \to M(A)$ a *-homomorphism: writing $m = (l_m, r_m^{\mathrm{op}})$, we define $m^* = (l_{m^*}, r_{m^*}^{\mathrm{op}})$ where

$$l_{m^*}(a) = (r_m(a^*))^*$$

and

$$r_{m^*}(a) = (l_m(a^*))^*.$$

Definition 2.1.7. *Let A, B be non-degenerate algebras. We say that a homomorphism $f : A \to M(B)$ has the* unique extension property *(or is* u.e. *(uniquely extendable)) if there exists an idempotent $p \in M(B)$ such that*

$$f(A) \cdot B = pB, \qquad B \cdot f(A) = Bp.$$

We then say that f has the extension property with respect to p. *We say that f* has the unique unital extension property *(or is* u.u.e. *(unital uniquely extendable)) if it has the unique extension property with respect to 1_B.*

The notion of 'being u.u.e.' appears in the appendix of [92], where however it is called *non-degeneracy* of the map f. It is also shown there that u.u.e. homomorphisms can be extended canonically to the multiplier algebra. The same holds for the more general notion of an u.e. homomorphism. First observe that if f is such a homomorphism, then it has the unique extension property with respect to a *unique* idempotent $p \in M(B)$. Then we can define $f(m)$ for $m \in M(A)$ to be the unique multiplier of B such that

$$f(m)(f(a)b) = f(ma)b$$

and

$$(bf(a))f(m) = bf(am)$$

for $a \in A$ and $b \in B$, and further

$$f(m)((1_B - p)b) = (b(1_B - p))f(m) = 0$$

for all $b \in B$. It is easily seen that this extension $f : M(A) \to M(B)$ (which should really be written $M(f)$) is then a homomorphism, sending 1_A to p.

Hence if the original f is in fact u.u.e., then this extension will be unital. Note however that not every unital homomorphism $f : M(A) \to M(B)$ necessarily restricts to an u.u.e. homomorphism $A \to M(B)$. Also remark that not every $f : A \to B$ which has a unique unit-preserving extension $M(A) \to M(B)$ necessarily has the unique unital extension property in our sense (consider a non-idempotent non-degenerate algebra and its identity map): one should rather regard the term 'unital' as referring to the range algebra B as a left and right A-module.

Lemma 2.1.8. *Let A, B, C be three non-degenerate algebras. Let $f : A \to M(B)$ and $g : B \to M(C)$ be u.e. homomorphisms, resp. with respect to idempotents $p \in M(B)$ and $q \in M(C)$. Then $g \circ f : A \to M(C)$ is u.e., with respect to the idempotent $g(p)q$.*

Proof. First note that $g(p)q$ is an idempotent since $q = g(1_B)$, hence commutes with $g(p)$. Then

$$\begin{aligned}(g \circ f)(A) \cdot C &= g(f(A)p)qC \\ &= g(f(A))g(p)qC,\end{aligned}$$

and similarly on the other side. □

Lemma 2.1.9. *Let A_1, A_2, B_1 and B_2 be four non-degenerate algebras. Let $f : A_1 \to M(B_1)$ and $g : A_2 \to M(B_2)$ be two u.e. homomorphisms, with respect to the respective idempotents $p_1 \in M(B_1)$ and $p_2 \in M(B_2)$. Then the homomorphism*

$$f \otimes g : A_1 \odot A_2 \to M(B_1) \odot M(B_2) \hookrightarrow M(B_1 \odot B_2)$$

is u.e. with respect to the idempotent $p_1 \otimes p_2$.

Proof. We have already remarked that the tensor product of non-degenerate algebras is again non-degenerate. Then the rest of the lemma is trivial to check:

$$\begin{aligned}(f(A_1) \odot f(A_2))(B_1 \odot B_2) &= f(A_1)B_1 \odot f(A_2)B_2 \\ &= p_1 B_1 \odot p_2 B_2 \\ &= (p_1 \otimes p_2) \cdot (B_1 \odot B_2),\end{aligned}$$

and similarly on the other side.

□

One can also define a notion of being (u.)u.e. for *anti*-multiplicative maps $A \to M(B)$ for A, B non-degenerate algebras. Then it is immediately verified that the tensor product of (u.)u.e. anti-multiplicative maps is again (u.)u.e. *However*, we can not *mix* an (u.)u.e. multiplicative with an anti-multiplicative maps in this way: if $f : A_1 \to M(B_1)$ is u.u.e. multiplicative, and $g : A_2 \to M(B_2)$ u.u.e. anti-multiplicative, there will in general be no well-defined linear map $A_1 \odot A_2 \to M(B_1 \odot B_2)$. For example, consider the case $A_i = B_1 = A$ a non-degenerate algebra, $B_2 = A^{\mathrm{op}}$, and f the identity map, g the canonical map $^{\mathrm{op}}$. Then if $m \in M(A \odot A)$, one can in general not interpret it as an element in $M(A \odot A^{\mathrm{op}})$, for then we would have to know if also $(1 \otimes a)m(a' \otimes 1) \in A \odot A$ for all $a, a' \in A$, which is not always the case. However, one can extend such a tensor product to a certain subalgebra of $M(A \odot A^{\mathrm{op}})$.

Definition 2.1.10. *Let A, B be non-degenerate algebras. We call* restricted multiplier tensor algebra *for A and B the space $M_{1;2}(A \odot B) \subseteq M(A \odot B)$ of multipliers such that $m(1_A \otimes b)$, $m(a \otimes 1_B)$, $(1_A \otimes b)m$ and $(a \otimes 1_B)m$ are elements of $A \odot B$, for all $a \in A$ and $b \in B$.*

More generally, if A_i is a finite collection of n non-degenerate algebras, we can introduce the space

$$M_{i_{11},i_{12},\ldots i_{1t_1};i_{21},i_{22},\ldots,i_{2t_2};\ldots;i_{s1},i_{s2},\ldots,i_{st_s}}(A_1 \odot A_2 \odot \ldots \odot A_n)$$

of multipliers m inside $M(A_1 \odot A_2 \odot \ldots \odot A_n)$, such that if we take, for any fixed k, the tensor product algebra of all $M(A_{i_{lr}})$ for $l \neq k$ and all $A_{i_{kr}}$, in the proper order, then this algebra, multiplied to either side of the element m, ends up in $A_1 \odot A_2 \odot \ldots \odot A_n$.

In any case, it is easy to check now that if A_i and B_i are non-degenerate algebras, and $f : A_1 \to M(B_1)$ is an (u.)u.e. homomorphism and $g : A_2 \to M(B_2)$ an (u.)u.e. anti-homomorphism, then $f \odot g$ can be extended to a linear map $M_{1;2}(A_1 \odot A_2) \to M(B_1 \odot B_2)$ in a unique way.

2.2 Morita theory for non-unital algebras

Definition 2.2.1. *Let A be an algebra, and V a left A-module.*

- *We call V* non-degenerate *if $a \cdot v = 0$ for all $a \in A$ implies $v = 0$.*

- *We call V* firm *when the map*

$$A \underset{A}{\odot} V \to V : a \underset{A}{\otimes} v \to a \cdot v$$

is bijective.

It is easy to see that the notion of unitality is weaker than that of firmness. Again, it is especially the notion of firmness which has been studied in the categorical framework.

We now introduce the notion of a linking algebra in the framework of non-unital algebras.

Definition 2.2.2. *A* linking algebra *is a couple (E, e) consisting of an algebra E, together with an idempotent $e \in M(E)$, such that e and $(1_{M(E)} - e)$ are* full*: $EeE = E$ and $E(1_{M(E)} - e)E = E$.*

We call a linking algebra firm, non-degenerate or 'with local units', whenever the underlying algebra has this property.

A linking *-algebra *is a linking algebra (E, e) such that E is a *-algebra and $e^* = e$.*

Note that by its definition, the algebra underlying a linking algebra is automatically idempotent.

We can still write E as a direct sum $\sum \oplus E_{ij}$, and we will also continue to write this direct sum in matrix form and its constituents by letters when convenient. Note that inside a linking algebra, the E_{ii} are automatically idempotent algebras, and all module structures on the E_{ij} are unital. Also note that when (E, e) is a linking *-algebra, the E_{ii} are *-algebras.

Similarly, one can introduce the non-unital versions of linking algebras between idempotent algebras, and we omit the obvious definition.

We leave it as an exercise to check that if (E, e, Φ_A, Φ_D) is a linking algebra between which is firm, or non-degenerate, or with local units, then both A and D have the same property. Also, if A and D are algebras with local units, then any linking algebra E between them also has. However, the fact that A and D are non-degenerate does not imply that a linking algebra between them is non-degenerate, and neither does the fact that A and D are firm imply that a linking algebra is firm, as the following example shows:

Example 2.2.3. *Denote by B the algebra A of Example 2.1.2, and by A the algebra of Example 2.1.3. Then* $E_1 = \begin{pmatrix} A & B \\ A & A \end{pmatrix}$ *and* $E_2 = \begin{pmatrix} B & A \\ B & B \end{pmatrix}$ *are examples of respectively a degenerate linking algebra between non-degenerate algebras, and a non-firm linking algebra between firm algebras.*

Proof. It is clear that E_1 and E_2 are well-defined since both are quotient algebras of $\begin{pmatrix} B & B \\ B & B \end{pmatrix}$, the first by dividing out the 2-sided ideal spanned by $\begin{pmatrix} \alpha^1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ \alpha^1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & \alpha^1 \end{pmatrix}$, the second by dividing out the 2-sided ideal spanned by $\begin{pmatrix} 0 & \alpha^1 \\ 0 & 0 \end{pmatrix}$.

It is also trivial to see that E_1 and E_2 are indeed linking algebras between resp. A and itself, and B and itself.

We already know that A is a non-degenerate algebra. However, E_1 is degenerate, since $\begin{pmatrix} 0 & \alpha^1 \\ 0 & 0 \end{pmatrix}$ is a zero multiplier.

We further know that B is firm. But the same argument as in Example 2.1.3 shows that E is not firm, by considering the element $\alpha_{11}^{1/2} \underset{E}{\otimes} \alpha_{12}^{1/2}$, where α_{kj}^i is the element α^i at position kj. □

Definition 2.2.4. *Let A and D be two idempotent algebras. We call them* Morita equivalent *when there exists a linking algebra between them.*

When A and D are firm (resp. non-degenerate and idempotent), we call them firmly *(resp.* non-degenerately*)* Morita equivalent *when there exists a firm (resp. non-degenerate) linking algebra between them.*

It is clear why we restrict to idempotent algebras in the first place: otherwise an algebra is not necessarily Morita equivalent with itself. However, even in the case of idempotent algebras, it is not immediately clear if this Morita equivalence really defines an equivalence relation. But it is easy to check that one can still define an identity linking algebra, the inverse of a linking algebra, and the composite of two linking algebras, in exactly the same way as for unital algebras, which clearly suffices to show that our Morita equivalence is an equivalence relation.

For firm algebras and firm linking algebras, it is then not difficult to show that the identity linking algebra provides a unit for composition (up to isomorphism), and the inverse an inverse for composition (up to isomorphism). This will not be true in general. However, for non-degenerate linking algebras, we can define the composition in a different way, which *will* make these two statements true. Namely, let E_{11}, E_{22} and E_{33} be idempotent non-degenerate algebras, $(E_1, e) = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix}$ a non-degenerate linking algebra between E_{11} and E_{22}, and $(E_2, e') = \begin{pmatrix} E_{22} & E_{23} \\ E_{32} & E_{33} \end{pmatrix}$ a non-degenerate linking algebra between E_{22} and E_{33}. Define

$$\pi_{ij}^k : E_{ij} \to \mathrm{Hom}_k(E_{jk}, E_{ik}) : z_{ij} \to (w_{jk} \to z_{ij} \cdot w_{jk}).$$

Then all π_{ij}^k are faithful, by the non-degeneracy of E_1 and E_2. So identifying E_{ij} with its image under π_{ij}^2, and defining $E_{13} := E_{12} \cdot E_{23}$ and $E_{31} = E_{32} \cdot E_{21}$ by composition of linear maps, it is easy to see that $\begin{pmatrix} E_{11} & E_{13} \\ E_{31} & E_{33} \end{pmatrix}$ is a linking algebra between E_{11} and E_{33}, which we shall then call the *composition* of E_2 with E_1. As for unital algebras, we will call $(E_{ij})_{i,j \in \{1,2,3\}}$ the associated 3×3-linking algebra between E_1 and E_2 (and similarly of course for the composition of firm linking algebras).

Lemma 2.2.5. *The composition of two non-degenerate linking algebras is again non-degenerate.*

Proof. Left non-degeneracy is easy to verify, using the linking algebra properties of E_1 and E_2, and the fact that E_{13} is *defined* by linear transformations from E_{32} to E_{12}.

Right non-degeneracy then also follows straightforwardly. For example, suppose $x_{12,i} \in E_{12}$ and $y_{23,i} \in E_{23}$ satisfy $\sum_i z_{11} x_{12,i} y_{23,i} = 0$ for all $z_{11} \in E_{11}$. Multiplying to the right with some $w_{32} \in E_{32}$, we find, since E_{12} is a non-degenerate left E_{11}-module, that $\sum_i x_{12,i} y_{23,i} w_{32} = 0$. Since w_{32} was arbitrary, $\sum_i x_{12,i} y_{23,i} = 0$.

□

For algebras with local units, it is easy to see that both possible compositions of linking algebras, either considering them as firm or non-degenerate, coincide, for the canonical map $E_{12} \underset{E_{22}}{\odot} E_{23} \to E_{13}$ is easily seen to be an isomorphism of E_{11}-E_{33}-linking algebras: if $x_{12,i} \in E_{12}$ and $y_{23,i} \in E_{23}$ with $\sum_i x_{12,i}y_{23,i} = 0$, choose a joint left local unit for the $x_{12,i}$, and write it in the form $\sum_j w_{12,j}z_{21_j}$. Then

$$\begin{aligned}\sum_i x_{12,i} \underset{E_{22}}{\otimes} y_{23,i} &= \sum_{i,j} (w_{12,j}z_{21_j}x_{12,i}) \underset{E_{22}}{\otimes} y_{23,i} \\ &= \sum_{i,j} w_{12,j} \underset{E_{22}}{\otimes} (z_{21,j}x_{12,i}y_{23,i}) \\ &= 0.\end{aligned}$$

Let (E, e) be a linking algebra. Then if $m \in eM(E)e$ (resp. $m \in (1_{M(E)} - e)M(E)(1_{M(E)} - e)$), we can restrict m to a multiplier of $A \subseteq E$ (resp. $D \subseteq E$).

Lemma 2.2.6. *Let (E, e) be a non-degenerate linking algebra. Then the natural map from $eM(E)e$ to $M(A)$ is an isomorphism. Similarly, $(1_E - e)M(E)(1_E - e)$ can be identified with $M(D)$.*

Proof. Consider the map

$$M(A) \to \mathrm{End}_k(C) : m \to (a \cdot c \to m \cdot (a \cdot c) := (ma) \cdot c).$$

This will be well-defined for the following reason: by unitality of C as left A-module, every element of C can be written as $\sum_i a_i \cdot c_i$, and if $\sum_i a_i \cdot c_i$ would happen to be zero, then

$$\begin{aligned}a \cdot (\sum_i (ma_i) \cdot c_i) &= \sum_i (am) \cdot (a_i \cdot c_i) \\ &= 0,\end{aligned}$$

and by the same calculation, also $b \cdot (\sum_i (ma_i) \cdot c_i) = 0$ for $b = \sum b_j \cdot a'_j \in B$. By unitality of B as a right A-module, *any* element of B can be written in this way, and so we find that

$$x \cdot \begin{pmatrix} 0 & 0 \\ \sum_i (ma_i) \cdot c_i & 0 \end{pmatrix}, \qquad \text{for all } x \in E,$$

so that also $\sum_i (ma_i) \cdot c_i = 0$ by non-degeneracy of E. This shows the well-definedness of the map. Similarly, we can define

$$M(A) \to \mathrm{End}_k(B) : m \to (b \cdot a \to b \cdot (am)).$$

Now if $m \in M(A)$, we can define a multiplier $\begin{pmatrix} 0 & 0 \\ 0 & m \end{pmatrix}$ in $eM(E)e$, determined in the obvious way, using the action of $M(A)$ on B and C introduced in the previous paragraph. It is clear that this will give us an inverse for the map whose definition was given just before the lemma.

The statement about D of course follows by symmetry.

□

By the previous lemma, we can unambiguously introduce the notations

$$M(B) := (1_E - e)M(E)e,$$

$$M(C) := eM(E)(1_E - e)$$

for a non-degenerate linking algebra (E, e). The same can of course be done for non-degenerate 3×3-linking algebras.

2.3 Multiplier Hopf algebras

The following definition was introduced in [92]. The notation used is explained at the end of section 2.1.

Definition 2.3.1. *A* multiplier Hopf algebra[2] *consists of a triple (A, M_A, Δ_A), with (A, M_A) an idempotent non-degenerate algebra, Δ_A a u.u.e. homomorphism $A \to M_{1;2}(A \odot A)$, called the* comultiplication *or* coproduct*, such that*

- $(\Delta_A \otimes \iota_A)\Delta_A = (\iota_A \otimes \Delta_A)\Delta_A$ *(coassociativity)*
- *the maps*

$$T_{\Delta_A,2} : A \odot A \to A \odot A : a \otimes a' \to \Delta_A(a)(1 \otimes a'),$$
$$T_{1,\Delta_A} : A \odot A \to A \odot A : a \otimes a' \to (a \otimes 1)\Delta_A(a'),$$
$$T_{\Delta_A,1} : A \odot A \to A \odot A : a \otimes a' \to \Delta_A(a)(a' \otimes 1),$$
$$T_{2,\Delta_A} : A \odot A \to A \odot A : a \otimes a' \to (1 \otimes a)\Delta_A(a')$$

  are bijective.

[2] Warning: What we define as a multiplier Hopf algebra, is called a *regular* multiplier Hopf algebra in [92].

Note that the first condition makes sense, since both Δ_A and ι_A, by idempotency of A, are u.u.e. maps, hence $(\Delta_A \otimes \iota_A)$ and $(\iota_A \otimes \Delta_A)$, which are u.u.e. maps $A \odot A \to M(A \odot A \odot A)$ by Lemma 2.1.9, can be extended to $M(A \odot A)$. Then we can also make sense of

$$\Delta_A^{(2)} := (\Delta_A \otimes \iota_A)\Delta_A = (\iota_A \otimes \Delta_A)\Delta_A$$

as a homomorphism $A \to M(A \odot A \odot A)$ (and in fact as a homomorphism $A \to M_{1,2;2,3;1,3}(A \odot A \odot A)$). We also remark that the bijectivities of the four T-maps are not all independent: for example, any one of them follows from the bijectivity of the other three.

We want to remark that the idempotency of A can in fact be dropped from the definition, by formulating the coassociativity condition in a slightly more complicated way (see the original article [92]). Then since the comultiplication is u.u.e., the surjectivity of one of the T-maps gives us that $A \odot A = A \odot A^2$ (or $A^2 \odot A$), from which the idempotency easily follows. (We should remark however that in the original article, also the u.u.e. property of Δ_A is dropped. This is no problem, since the surjectivity of the T-maps implies $(A \odot A)\Delta_A(A) = \Delta_A(A)(A \odot A) = A \odot A^2 = A^2 \odot A$. Since $A^2 \neq 0$ by non-degeneracy of A, this implies $A^2 = A$ and hence also Δ_A u.u.e.)

The following result comes from [92]. The techniques used for proving this statement have in fact already made their appearance in the first chapter, and will later be used again, so we do not provide the proofs.

Proposition 2.3.2. *Let A be a multiplier Hopf algebra. Then there exists a unique linear map $\varepsilon_A : A \to k$, called the* counit*, such that*

$$(\varepsilon_A \otimes \iota_A)(\Delta_A(a)(1 \otimes a')) = a \cdot a',$$

$$(\iota_A \otimes \varepsilon_A)(\Delta_A(a)(a' \otimes 1)) = a \cdot a'.$$

This ε_A will then be a homomorphism.

There also exists a unique linear map $S_A : A \to A$, called the antipode*, such that*

$$M_A((S_A \otimes \iota_A)(\Delta_A(a)(1 \otimes a'))) = \varepsilon_A(a)a',$$

$$M_A((\iota_A \otimes S_A)((a' \otimes 1)\Delta_A(a))) = \varepsilon_A(a)a'.$$

This map will then be an anti-comultiplicative anti-automorphism.

In fact, multiplier Hopf algebras can also be defined by asking the existence of antipode and counit instead of the bijectivity of the four T-maps, as is more customary in the Hopf algebra case. However, the definition we gave was the original one, and is also the one which reappears most naturally in the analytic framework.

We will still use Sweedler notation for a multiplier Hopf algebra A, that is, write $\Delta_A(a)$ as $a_{(1)} \otimes a_{(2)}$, but now this expression is even more formal than for Hopf algebras, because $a_{(1)} \otimes a_{(2)}$ doesn't even denote a sum of simple tensor elements. For example, the expression $a_{(1)}a_{(2)}a_{(3)}a_{(4)}$ will in general be meaningless. However, when one of the legs of $\Delta_A(a)$ is covered, for example, when the first leg of $\Delta_A(a)$ is covered by a' to the left, as in $(a' \otimes 1)\Delta_A(a)$, then the expression $a' \cdot a_{(1)} \otimes a_{(2)}$ *does* become simply a finite sum of elementary tensors. We refer to [96] for a careful analysis of this technique.

If A is a multiplier algebra, and ω is a functional $A \to k$, one can make sense of $(\iota_A \otimes \omega)(\Delta_A(a))$ as a multiplier of A, by

$$(\iota_A \otimes \omega)(\Delta_A(a))a' := (\iota_A \otimes \omega)(\Delta_A(a)(a' \otimes 1)),$$

$$a'(\iota_A \otimes \omega)(\Delta_A(a)) := (\iota_A \otimes \omega)((a' \otimes 1)\Delta_A(a)).$$

Similarly, $(\omega \otimes \iota_A)(\Delta_A(a))$ is a multiplier of A.

We also remark the following nice property of the underlying algebra of a multiplier Hopf algebra (see [29]):

Proposition 2.3.3. *Let A be a multiplier Hopf algebra. Then A has local units.*

2.4 Algebraic and *-algebraic quantum groups

2.4.1 Algebraic quantum groups

Multiplier Hopf algebras become especially nice when they possess a certain special functional. The following definition comes from [93].

Definition 2.4.1. *An* algebraic quantum group *is a multiplier Hopf algebra A for which there exists a non-zero functional $\varphi_A : A \to k$ such that*

$$(\iota_A \otimes \varphi_A)\Delta_A(a) = \varphi_A(a)1_A \qquad \textit{for all } a \in A.$$

Such a φ_A is called a left invariant functional.

Here are some nice facts about left invariant functionals (we refer to [93] for proofs):

Definition-Proposition 2.4.2. *Let A be an algebraic quantum group, and φ_A a left invariant functional.*

1. *If φ'_A is another left invariant functional, then $\varphi'_A = \lambda \cdot \varphi_A$ for some non-zero $\lambda \in k$.*

2. *The functional φ_A is faithful: if $a \in A$ and $((\varphi_A(aa') = 0$ for all $a' \in A)$ or $(\varphi_A(a'a) = 0$ for all $a' \in A))$, then $a = 0$.*

3. *There exists a unique automorphism σ_A of A, called the* modular automorphism of φ_A*, such that*
$$\varphi_A(a'\sigma_A(a)) = \varphi_A(aa') \qquad \text{for all } a, a' \in A,$$
and then $\varphi_A \circ \sigma_A = \varphi_A$.

4. *The functional $\psi_A := \varphi_A \circ S_A$ is right invariant:*
$$(\psi_A \otimes \iota_A)(\Delta_A(a)) = \psi_A(a)1_A \qquad \text{for all } a \in A.$$

5. *Again with $\psi_A = \varphi_A \circ S_A$, there exists a unique invertible multiplier $\delta_A \in M(A)$, called* the modular element of A*, such that*
$$\psi_A(a) = \varphi_A(a\delta_A)$$
and
$$(\varphi_A \otimes \iota_A)(\Delta_A(a)) = \varphi_A(a)\delta_A$$
for all $a \in A$. Moreover, $\varsigma_A(a) := \delta_A\sigma_A(a)\delta_A^{-1}$ is then a modular automorphism for ψ_A.

6. *There exists a non-zero number $\nu_A \in k$, called* the scaling constant of A*, such that $\varphi_A \circ S_A^2 = \nu_A \cdot \varphi_A$ and $\sigma_A(\delta_A) = \nu_A^{-1}\delta_A$.*

7. *The following commutation relations hold:*
$$S_A \circ \sigma_A = \varsigma_A^{-1} \circ S_A,$$
$$S_A \circ \varsigma_A = \sigma_A^{-1} \circ S_A,$$
$$\Delta_A \circ S_A^2 = (\sigma_A \otimes \varsigma_A^{-1}) \circ \Delta_A,$$
$$\Delta_A \circ \sigma_A = (S_A^2 \otimes \sigma_A) \circ \Delta_A,$$
$$\Delta_A \circ \varsigma_A = (\varsigma_A \otimes S_A^{-2}) \circ \Delta_A.$$

The nicest thing about algebraic quantum groups is that they allow for a duality theory. Note that by the previous proposition, the following vector spaces are equal:

$$\{\varphi_A(\cdot\, a) \mid a \in A\},$$
$$\{\varphi_A(a\, \cdot) \mid a \in A\},$$
$$\{\psi_A(\cdot\, a) \mid a \in A\},$$
$$\{\psi_A(a\, \cdot) \mid a \in A\}.$$

We denote by $\widehat{A}$ this canonical subspace of A^*, the dual vector space of A, and call it the *restricted dual* of A. The functional $\omega \in \widehat{A}$, evaluated in $a \in A$, will be denoted as $\omega(a)$, although sometimes, when we view A as a subspace of $(\widehat{A})^*$ in a natural way, we also denote it as $a(\omega)$.

We can equip $\widehat{A}$ with a non-degenerate multiplication $M_{\widehat{A}}$: for $\omega_1, \omega_2 \in \widehat{A}$, their product is defined to be the functional

$$(\omega_1 \cdot \omega_2)(a) = \omega_1((\iota \otimes \omega_2)(\Delta_A(a)),$$

which is meaningful by the precise form of the ω_i. One then shows that this product ends up in $\widehat{A}$. We can also equip $\widehat{A}$ with a u.u.e. coassociative comultiplication $\Delta_{\widehat{A}}$, turning it into a multiplier Hopf algebra. This $\Delta_{\widehat{A}}$ is uniquely determined by

$$(\Delta_{\widehat{A}}(\omega_1)(1_{\widehat{A}} \otimes \omega_2))(a' \otimes a) := \omega_1(a' a_{(1)})\omega_2(a_{(2)}).$$

Finally, $\widehat{A}$ is an algebraic quantum group, a left invariant functional $\varphi_{\widehat{A}}$ being given by the formula

$$\varphi_{\widehat{A}}(\omega) = \varepsilon(a) \text{ when } \omega = \varphi_A(a\, \cdot).$$

(Note that by faithfulness of φ_A, such an a is uniquely determined.) The dual of $\widehat{A}$ is then canonically isomorphic to A as an algebraic quantum group, by sending a to the functional $a(\,\cdot\,)$. One can then also directly interpret $M(\widehat{A} \odot \widehat{A})$ as a subspace of $(A \odot A)^*$, and the formula for the comultiplication simplifies to

$$\Delta_{\widehat{A}}(\omega)(a \otimes a') = \omega(a \cdot a').$$

Since one can interpret $M(\widehat{A})$ as functionals on A, we can ask ourselves how the functional $\delta_{\widehat{A}}$ looks like. This, and similar questions, are answered by the following Proposition:

Proposition 2.4.3. *Let A be an algebraic quantum group, and $\widehat{A}$ its dual. Let $\delta_{\widehat{A}}$ be the modular element of $\widehat{A}$, and σ_A the modular automorphism of φ_A. Then for all $a \in A$, we have*

$$\delta_{\widehat{A}}(a) = \varepsilon_A(\sigma_A^{-1}(a)).$$

Further, we have

$$(\sigma_{\widehat{A}}(\omega))(a) = \omega(S_A^2(a)\delta_A^{-1}),$$
$$(\sigma'_{\widehat{A}}(\omega))(a) = \omega(\delta_A^{-1}S_A^{-2}(a)).$$

The result concerning $\delta_{\widehat{A}}$ was noted in [55], and a straightforward algebraic proof (in a more general setting) was given in [25].

2.4.2 *-Algebraic quantum groups

The following *-version of multiplier Hopf algebras and algebraic quantum groups was also given in [92] and [93].

Definition 2.4.4. *A* multiplier Hopf *-algebra *is a multiplier Hopf algebra over $\mathbb{C}$, together with a *-algebra structure on the underlying algebra, in such a way that $\Delta_A(a^*) = \Delta_A(a)^*$.*

A *-algebraic quantum group *A is an algebraic quantum group over $\mathbb{C}$, which is at the same time a multiplier Hopf *-algebra,* such that *there exists a* positive *left invariant functional φ_A:*

$$\varphi_A(a^*a) \geqslant 0 \quad \textit{for all } a \in A.$$

We note how the *-structure of A interacts (in both cases) with the other structure of A (see [93]): we have that ε_A is a *-homomorphism and that $S_A(a^*) = S_A^{-1}(a)^*$, and in the case of *-algebraic quantum groups, we have that $\delta_A^* = \delta_A$, and $\sigma_A(a^*) = \sigma_A^{-1}(a)^*$ and $\sigma'_A(a^*) = \sigma'^{-1}_A(a)^*$. Also, in this last case $\widehat{A}$ will then be a *-algebraic quantum group, with *-structure $\omega^*(a) = \overline{\omega(S_A(a)^*)}$. The formula for a left invariant functional on $\widehat{A}$, given in the previous section, will automatically give us a *positive* functional.

The condition of positivity on the left invariant functional is a strong one: for example, in the definition, one only has to ask the non-degeneracy of the underlying algebra to deduce the complete positivity of the underlying *-algebra, for then already $\varphi_A(a^*a) = 0$ will imply $a = 0$. This follows almost straightforwardly from the Cauchy-Schwarz inequality and the faithfulness of φ_A as a functional on the associated algebra, *except* that we also need to use the existence of local units in A to know that φ_A is hermitian (i.e. $\varphi_A(a^*) = \overline{\varphi_A(a)}$). It is then of course obvious that A, with the inner product $\langle a, a' \rangle_A = \varphi_A(a'^* a)$, becomes a pre-Hilbert space.

The positivity of φ_A also allows us to put an analytic structure on a *-algebraic quantum group, and to fit it into the theory of locally compact quantum groups (which is recalled in the fifth chapter of this thesis). This was first observed in [55], but the methods used were highly non-trivial, and relied on some heavy machinery. In [21] it was observed that these results could be arrived at in a much simpler way, without even leaving the realm of pure algebra. Moreover, it tells a lot more about the actual structure of *-algebraic quantum groups. We reproduce these results here.

Lemma 2.4.5. *Let A be a *-algebraic quantum group. If a is a non-zero element in A and n is an even integer, then $a^*((\sigma_A)^n S_A^{2n})(a) \neq 0$.*

Proof. Suppose that $a \in A$ and $n \in 2\mathbb{Z}$ are such that

$$a^*((\sigma_A)^n S_A^{2n}(a)) = 0.$$

Then using that σ_A and S_A^2 commute, applying $\sigma_A^{-n/2} S_A^{-n}$ and using the commutation with *, we find that

$$(\sigma_A^{n/2} S_A^n(a))^* (\sigma_A^{n/2} S_A^n(a)) = 0.$$

Since A is positive, $\sigma_A^{n/2} S_A^n(a) = 0$, hence $a = 0$.

□

Lemma 2.4.6. *Let A be a *-algebraic quantum group, and write $\kappa_A = \sigma_A^{-1} S_A^2$. If $a \in A$, then the linear span of the $\kappa_A^n(a)$, with $n \in \mathbb{Z}$, is finite-dimensional.*

Proof. Let a be a fixed element of A. Choose a non-zero $b \in A$, and write

$$a \otimes b = \sum_{i=1}^{n} \Delta_A(p_i)(1 \otimes q_i),$$

with $p_i, q_i \in A$. Denote $\rho_A = \sigma_A S_A^2$. Then using the commutation relations of Definition-Proposition 2.4.2, we find

$$\kappa_A^n(a) \otimes \rho_A^{-n}(b) = \sum \Delta_A(p_i)(1 \otimes \rho_A^{-n}(q_i)), \qquad \text{for all } n \in \mathbb{Z}.$$

Multiply this equation to the left with $1 \otimes b^*$ to get

$$\kappa_A^n(a) \otimes b^* \rho_A^{-n}(b) = \sum ((1 \otimes b^*)\Delta_A(p_i))(1 \otimes \rho_A^{-n}(q_i)).$$

Choose $a_{ij}, b_{ij} \in A$ such that

$$(1 \otimes b^*)\Delta_A(p_i) = \sum_{j=1}^{m_i} a_{ij} \otimes b_{ij},$$

and let L be the finite-dimensional space spanned by the a_{ij}. We see that $\kappa_A^n(a) \otimes b^* \rho_A^{-n}(b) \in L \otimes A$, for every $n \in \mathbb{Z}$. Using the previous lemma, we can conclude that $\kappa_A^n(a) \in L$ for all $n \in 2\mathbb{Z}$. But this easily implies that the linear span of all $\kappa_A^n(a)$, with $n \in \mathbb{Z}$, is a finite-dimensional, κ_A-invariant linear subspace of A.

□

Now note that for $\omega \in \widehat{A}$ and $b \in A$, we have, by Proposition 2.4.3,

$$\begin{aligned}
(\omega \cdot \delta_{\widehat{A}})(b) &= (\varepsilon_A \circ \sigma_A^{-1})((\omega \otimes \iota_A)(\Delta_A(b))) \\
&= \varepsilon_A((\omega \circ S_A^2 \otimes \iota_A)(\Delta_A(\sigma_A^{-1}(b)))) \\
&= \omega(\kappa_A(b)),
\end{aligned}$$

where κ_A still denotes $\sigma_A^{-1} S_A^2$. If ω is of the form $\varphi_A(\,\cdot\, a)$, then

$$\begin{aligned}
(\varphi_A(\,\cdot\, a) \cdot \delta_{\widehat{A}})(b) &= \varphi_A(\sigma_A^{-1}(S_A^2(b))a) \\
&= \varphi_A(a S_A^2(b)) \\
&= \nu_A \varphi_A(S_A^{-2}(a) b) \\
&= \nu_A \varphi_A(b \kappa_A^{-1}(a)).
\end{aligned}$$

So the previous result implies that for ω fixed, the linear span of the $\omega \cdot \delta_{\widehat{A}}^n$ is finite-dimensional. The same is then also true for *left* multiplication with $\delta_{\widehat{A}}$.

By biduality, we conclude that for each a in A, the linear span of the $\delta_A^n \cdot a$ is a finite-dimensional space K. Since left multiplication with δ_A is a self-adjoint operator on K, with Hilbert space structure induced by φ_A (i.e. $\langle a, b \rangle_A := \varphi_A(b^* a)$), we can diagonalize δ_A. Hence we arrive at

Proposition 2.4.7. *Let A be a *-algebraic quantum group. Then A is spanned by elements which are eigenvectors for left multiplication by δ_A.*

We can use this to answer a question of [53]:

Theorem 2.4.8. *Let A be a *-algebraic quantum group. Then the scaling constant ν_A equals 1.*

Proof. Choose a non-zero element $a \in A$ with $\delta_A a = \lambda a$, for some $\lambda \in \mathbb{R}_0$. Then $\varphi_A(aa^*\delta_A) = \lambda\varphi_A(aa^*)$. But the left hand side equals

$$\nu_A^{-1}\varphi_A(\delta_A aa^*) = \nu_A^{-1}\lambda\varphi_A(aa^*).$$

Since $\varphi_A(aa^*) \neq 0$, we arrive at $\nu_A = 1$.

□

Proposition 2.4.7 can be strengthened:

Theorem 2.4.9. *Let A be a *-algebraic quantum group. Then A is spanned by elements which are simultaneously eigenvectors for S_A^2, σ_A and σ_A, and left and right multiplication by δ_A. Moreover, the eigenvalues of these actions are all positive.*

Proof. We know that A is spanned by eigenvectors for left multiplication with δ_A. The same is then true for right multiplication with δ_A, using that right multiplication with δ_A is still self-adjoint with respect to $\langle \cdot, \cdot \rangle_A$, using that $\delta_A^* = \delta_A = \sigma_A(\delta_A)$ by the previous theorem. The eigenvectors of $\kappa_A = \sigma_A^{-1}S_A^2$ and $\rho_A = \sigma_A S_A^2$ then also span A, since these are easily shown to be self-adjoint operators with respect to the natural Hilbert space structure on A, and since we have moreover shown in Lemma 2.4.6 that A is the union of finite-dimensional globally invariant subspaces for them. Since all these operations commute, we can find a basis of A consisting of simultaneous eigenvectors. Since σ_A, σ_A and S_A^2 can be written as compositions of the maps κ_A, ρ_A and left and right multiplication with δ_A, the first part of the theorem is proven.

We show that left multiplication with δ_A has positive eigenvalues. Fix $a \in A$. Let λ be an eigenvalue for left multiplication with δ_A, and b an eigenvector for it. Consider $c = \Delta(a)(1 \otimes b)$. Then $(\varphi_A \otimes \varphi_A)(c^*c)$ will be a positive number. But this is equal to $\varphi_A(a^*a)\varphi_A(b^*\delta_A b) = \lambda\varphi_A(a^*a)\varphi_A(b^*b)$. Hence λ must be positive. Then also right multiplication with δ_A will have positive eigenvalues. As before, duality implies that κ_A and ρ_A have positive eigenvalues (cf. the discussion before Proposition 2.4.7), hence the same is true of σ_A, σ_A and S_A^2.

□

This theorem *explains* why there exists an analytic structure on a *-algebraic quantum group A (cf. [55]).

We can also see now that $\psi_A = \varphi_A \circ S_A$ is a positive right invariant functional, which is a priori not clear. Indeed: we can define a multiplier $\delta_A^{1/2}$ in $M(A)$, by the unique property that if $a \in A$ is an eigenvector for left multiplication with δ_A with eigenvalue λ, then $\delta_A^{1/2} \cdot a = \lambda^{1/2} a$. Then $(\delta_A^{1/2})^* = \delta_A^{1/2}$ and $(\delta_A^{1/2})^2 = \delta_A$. An easy eigenvector argument also shows that $\sigma_A(\delta_A^{1/2}) = \delta_A^{1/2}$. Hence

$$\begin{aligned} \psi_A(a^*a) &= \varphi_A(a^*a\delta_A) \\ &= \varphi_A((a\delta_A^{1/2})^* a\delta_A^{1/2}) \\ &\geqslant 0, \end{aligned}$$

and ψ_A is positive.

2.5 Galois coactions for multiplier Hopf algebras

We introduce some definitions and results concerning (Galois) coactions for multiplier Hopf algebras, taken from [97]. We follow again the notation used at the end of section 2.1.

Definition 2.5.1. *Let A be a multiplier Hopf algebra, and let B be a non-degenerate algebra. A* right coaction α_B *of A on B is an injective u.u.e. homomorphism*

$$\alpha_B : B \to M_2(B \odot A)$$

satisfying $(\alpha_B \otimes \iota_A)\alpha_B = (\iota_B \otimes \Delta_A)\alpha_B$.

The defining property is meaningful since $(\alpha_B \otimes \iota_A)$ and $(\iota_B \otimes \Delta_A)$ are u.u.e. homomorphisms $B \odot A \to M(B \odot A \odot A)$, hence have unique extensions to homomorphisms $M(B \odot A) \to M(B \odot A \odot A)$. The maps $B \odot A \to B \odot A$ given by

$$\begin{aligned} T_{\alpha_B,2} &: b \otimes a \to \alpha_B(b)(1_B \otimes a), \\ T_{2,\alpha_B} &: b \otimes a \to (1_B \otimes a)\alpha_B(b) \end{aligned}$$

are then well-defined bijections, their inverses determined by

$$\begin{aligned} T^{-1}_{\alpha_B,2} &: b \otimes S_A(a) \to (\iota_B \otimes S_A)((1_B \otimes a)\alpha_B(b)), \\ T^{-1}_{2,\alpha_B} &: b \otimes S_A^{-1}(a) \to (\iota_B \otimes S_A^{-1})(\alpha_B(b)(1_B \otimes a)). \end{aligned}$$

Note then that since α_B is u.u.e., this says that $B \odot A = \alpha_B(B)(B \odot A) = B^2 \odot A$, so B is automatically idempotent.

The injectivity of α_B implies that $(\iota_B \otimes \varepsilon_A)(\alpha_B(b)) = b$ for all $b \in B$ (where a priori the left hand side has to be treated as a multiplier).

Definition 2.5.2. *Let A be a multiplier Hopf algebra, and α_B a coaction of A on a non-degenerate algebra B. Then α_B is called reduced if $\alpha_B(B) \subseteq M_{1;2}(B \odot A)$.*

In fact, one only has to ask that

$$(B \otimes 1_A)\alpha_B(B) \subseteq B \odot A,$$

for then automatically

$$\alpha_B(B)(B \otimes 1_A) \subseteq B \odot A$$

(see the remark after Proposition 2.5 in [97]).

Definition 2.5.3. *Let α_B be a coaction of a multiplier Hopf algebra A on a non-degenerate algebra B. The algebra of coinvariants $F = B^{\alpha_B} \subseteq M(B)$ for α_B is the unital algebra of elements b in $M(B)$ such that $\alpha_B(b) = b \otimes 1_A$.*

Definition 2.5.4. *Let A be a multiplier Hopf algebra, and α_B a coaction of A on B. We call α_B a* Galois coaction*, or say that it has the Galois property, if it is reduced, and if the map*

$$G : B \underset{B^{\alpha_B}}{\odot} B \to B \odot A : b \underset{F}{\otimes} b' \to (b \otimes 1_A)\alpha_B(b'),$$

which is called a Galois map *for α_B, is bijective.*

When A is an algebraic quantum group, the bijectivity of G in the previous definition already follows from the surjectivity of this map (see Theorem 4.4. in [97]). Also, for Galois coactions of general multiplier Hopf algebras, we have that α_B is Galois iff the map

$$H : B \underset{F}{\odot} B \to B \odot A : b \underset{F}{\otimes} b' \to \alpha_B(b)(b' \otimes 1_A)$$

is bijective. For example, in case G is bijective, the inverse map of H is given by

$$H^{-1}(b \otimes a) = G^{-1}((1 \otimes S_A^{-1}(a))\alpha_B(b)).$$

Chapter 3

Galois objects for algebraic quantum groups

In this chapter, we develop a theory of Galois objects for algebraic quantum groups, i.e. of Galois coactions with trivial algebra of coinvariants. The emphasis here is mainly on the structure of the Galois object themselves: we postpone the reflection technique, already encountered in the first chapter in the setting of Hopf algebras, to the fourth chapter. We show that Galois objects for algebraic quantum groups possess a faithful invariant functional, a modular automorphism for this functional, and also a modular element. We further show that they possess an analogue of the antipode squared of a quantum group. This latter map is defined in a way which is specifically adapted to the algebraic quantum group case, and there seems no analogue of this map for Galois objects for multiplier Hopf algebras, without imposing extra, not very natural conditions. We also consider the special cases of Galois objects for algebraic quantum groups of compact and discrete type, and for *-algebraic quantum groups.

3.1 Definition of Galois objects

Definition 3.1.1. *Let A be a multiplier Hopf algebra. A* right Galois object *(B,α_B) for A is a non-degenerate algebra B, with a right Galois coaction α_B of A on B,* such that *the algebra B^{α_B} of coinvariants equals $k \cdot 1_B$.*

We will also talk about right A-Galois objects B.

In this chapter, we will now say nothing about the general case of multiplier Hopf algebras, but will exclusively treat the case of Galois objects for algebraic quantum groups. Therefore, when talking about right Galois objects, we always mean right Galois objects *with respect to an algebraic quantum group*. For the rest, we keep using the notation as in the last section of the previous chapter.

Proposition 3.1.2. *Let B be a right A-Galois object. For $a \in A$, there exists a (unique) $\tilde{\beta}_A(a) \in M(B \odot B)$ which satisfies*

$$\begin{aligned}(b \otimes 1)\tilde{\beta}_A(a) &= G^{-1}(b \otimes a),\\ \tilde{\beta}_A(a)(1 \otimes b) &= H^{-1}(b \otimes S_A(a)).\end{aligned}$$

for all $b \in B$ and $a \in A$.

Proof. We have to see if

$$(G^{-1}(b \otimes a)) \cdot (1 \otimes b') = (b \otimes 1) \cdot (H^{-1}(b' \otimes S_A(a))).$$

Now $G(b \otimes b'b'') = G(b \otimes b') \cdot \alpha(b'')$, so

$$G^{-1}(b \otimes a) \cdot (1 \otimes b') = G^{-1}((b \otimes a)\alpha_B(b')).$$

Similarly, $H(bb' \otimes b'') = \alpha_B(b)H(b' \otimes b'')$, so

$$(b \otimes 1) \cdot (H^{-1}(b' \otimes S_A(a))) = H^{-1}(\alpha_B(b)(b' \otimes S_A(a))).$$

By the formula for H^{-1} given at the end of section 2.5, we then only have to see if

$$(b \otimes a)\alpha_B(b') = (T_{2,\alpha_B} \circ (\iota_B \otimes S_A^{-1}))(\alpha_B(b)(b' \otimes S_A(a))).$$

Since $T_{2,\alpha_B}(b \otimes aa') = (1 \otimes a) \cdot T_{2,\alpha_B}(b \otimes a')$, this reduces to proving that

$$(b \otimes 1)\alpha_B(b') = (T_{2,\alpha_B} \circ (\iota_B \otimes S_A^{-1}))(\alpha_B(b)(b' \otimes 1)).$$

But this says exactly that $G = T_{2,\alpha_B} \circ (\iota_B \otimes S_A^{-1}) \circ H$, which follows again by the identity at the end of section 2.5.

$\square$

We will show later that also the maps

$$b \otimes a \to \tilde{\beta}_A(a)(b \otimes 1)$$

and

$$b \otimes a \to (1 \otimes b)\tilde{\beta}_A(a)$$

are bijections from $B \odot A$ to $B \odot B$ (see Corollary 3.5.2). This will allow us to regard $\tilde{\beta}_A$ rather as a map $\beta_A : A \to M(B^{\mathrm{op}} \odot B)$, which is really the more natural viewpoint.

For computations we will keep using the Sweedler notation for Galois objects, denoting

$$\alpha_B(b) = b_{(0)} \otimes b_{(1)}$$

and

$$\tilde{\beta}_B(a) = a^{[1]} \otimes a^{[2]}.$$

Then by definition we have the identities

$$\begin{aligned} ba^{[1]}{a^{[2]}}_{(0)} \otimes {a^{[2]}}_{(1)} &= b \otimes a, \\ {a^{[1]}}_{(0)} a^{[2]} b \otimes {a^{[1]}}_{(1)} &= b \otimes S(a), \end{aligned}$$

for all $b \in B, a \in A$. Applying $\iota_B \otimes \varepsilon_A$ to the first equation, we obtain the formula

$$ba^{[1]}a^{[2]} = \varepsilon_A(a)b.$$

We want to remark and warn again that the use of the Sweedler notation here is more delicate than for Hopf algebras.

3.2 The existence of invariant functionals

For any functional ω on a right Galois object B, we can still interpret $(\omega \otimes \iota_A)(\alpha_B(b))$ in a natural way as a multiplier of A. On the other hand, by reducedness of the coaction α_B, we can also interpret $(\iota_B \otimes \omega)(\alpha_B(b))$ as a multiplier of B, for any $b \in B$ and $\omega \in A^*$.

Definition 3.2.1. *Let B be a right A-Galois object. By an* invariant functional *on B we mean a functional ω on B such that $(\omega \otimes \iota_A)(\alpha_B(b)) = \omega(b)1_A$ for all $b \in B$. More generally, if m is a multiplier of A, we mean by an m-*invariant functional *on B a functional ω on B such that $(\omega \otimes \iota_A)(\alpha_B(b)) = \omega(b)m$ for all $b \in B$.*

Theorem 3.2.2. *Let B be a right A-Galois object. There exists a faithful δ_A-invariant functional φ_B on B such that*

$$(\iota_B \otimes \varphi_A)(\alpha_B(b)) = \varphi_B(b)1_A$$

for all $b \in B$.

Recall that the faithfulness of the functional φ_B means that $\varphi_B(bb') = 0$ for all b' implies $b = 0$, as does $\varphi_B(b'b) = 0$ for all b'.

Proof. Take $b, b' \in B$ and $a \in A$. Denote b'' for $(\iota_B \otimes \varphi_A)(\alpha_B(b)) \in M(B)$. Then we compute in detail, using the definition of the extension of α_B to $M(B)$, of $(\alpha_B \otimes \iota_A)$ to $M(A \otimes A)$, and the defining left invariance property of φ_A:

$$\begin{aligned}
&\alpha_B(b'')(\alpha_B(b')(1 \otimes a)) \\
&\quad = \alpha_B(b''b')(1 \otimes a) \\
&\quad = \alpha_B((\iota_B \otimes \varphi_A)(\alpha_B(b)(b' \otimes 1)))(1 \otimes a) \\
&\quad = (\iota_B \otimes \iota_A \otimes \varphi_A)((\alpha_B \otimes \iota_A)(\alpha_B(b)(b' \otimes 1))(1 \otimes a \otimes 1)) \\
&\quad = (\iota_B \otimes \iota_A \otimes \varphi_A)((\alpha_B \otimes \iota_A)(\alpha_B(b))(\alpha_B(b') \otimes 1)(1 \otimes a \otimes 1)) \\
&\quad = (\iota_B \otimes \iota_A \otimes \varphi_A)((\iota_B \otimes \Delta_A)(\alpha_B(b))(b'_{(0)} \otimes b'_{(1)}a \otimes 1)) \\
&\quad = (\iota_B \otimes \iota_A \otimes \varphi_A)((\iota_B \otimes \Delta_A)(\alpha_B(b)(b'_{(0)} \otimes 1))(1 \otimes b'_{(1)}a \otimes 1)) \\
&\quad = b_{(0)}b'_{(0)} \otimes (\iota_A \otimes \varphi_A)(\Delta_A(b_{(1)})(b'_{(1)}a \otimes 1)) \\
&\quad = b_{(0)}b'_{(0)} \otimes \varphi_A(b_{(1)})b'_{(1)}a \\
&\quad = (b'' \otimes 1)(\alpha_B(y)(1 \otimes a)),
\end{aligned}$$

where the reader should make sure for himself that these expressions are all well-covered. It follows that $b'' = (\iota_B \otimes \varphi_A)(\alpha_B(b))$ is coinvariant, so $b'' = \varphi_B(b)1_B$ for some scalar $\varphi_B(b)$, by definition of a Galois object. It is clear that φ_B then defines a linear functional on B.

We show now that this map φ_B is δ_A-invariant: for $b, b' \in B$ and $a \in A$, we have

$$\begin{aligned}
\varphi_B(b_{(0)})b' \otimes b_{(1)}a &= b_{(0)}b' \otimes \varphi_A(b_{(1)})b_{(2)}a \\
&= b_{(0)}\varphi_A(b_{(1)})b' \otimes \delta_A a \\
&= \varphi_B(b)b' \otimes \delta_A a,
\end{aligned}$$

where we used that $(\varphi_A \otimes \iota_A)(\Delta_A(a)) = \varphi_A(a)\delta_A$ for $a \in A$.

Finally, we prove faithfulness. Suppose $b \in B$ is such that $\varphi_B(bb') = 0$ for all $b' \in B$. Then

$$\varphi_A(b_{(1)}b'_{(1)})b_{(0)}b'_{(0)}b'' = 0 \qquad \text{for all } b', b'' \in B.$$

Using the Galois property, it follows that

$$\varphi_A(b_{(1)}a)b_{(0)}b' = 0 \qquad \text{for all } b' \in B \text{ and } a \in A.$$

The faithfulness of φ_A implies that $b_{(0)}b' \otimes b_{(1)} = 0$ for all $b' \in B$, hence $b = 0$. Likewise $\varphi_B(b'b) = 0$ for all $b' \in B$ implies $b = 0$. □

Corollary 3.2.3. *The underling algebra B of a right A-Galois object has local units.*

Proof. One can copy for example the proof of Proposition 2.6 in [29]: let $b \in B$, and suppose $b \cdot B$ does not contain b. Then we can find $\omega \in B^*$ with $\omega(b) = 1$ but $\omega_{|b \cdot B} = 0$. Then also $\omega(bb'_{(0)})b'_{(1)}a = 0$ for all $b' \in B$ and $a \in A$. Hence $\omega(bb'_{(0)})b'_{(1)} = 0$ for all $b' \in B$, and applying φ_A, we get $\varphi_B(b')\omega(b) = 0$. Since φ_B is a non-zero functional, this is only possible if $\omega(b) = 0$, which gives a contradiction. □

Proposition 3.2.4. *Let B be a right A-Galois object. For $a \in A$ and $b \in B$, we have*

$$\varphi_B(a^{[2]})ba^{[1]} = \varphi_A(a)b$$

and

$$\varphi_B(a^{[1]})a^{[2]}b = \psi_A(a)b.$$

Proof. Using the explicit form for the inverses of the maps G and H, given in Proposition 3.1.2, the stated identities are equivalent to the identities $\varphi_A(b_{(1)})b'b_{(0)} = \varphi_B(b)b'$ and $(\psi_A \circ S_A^{-1})(b_{(1)})b_{(0)}b' = \varphi_B(b)b'$ for all $b, b' \in B$, which hold true by definition of φ_B.

□

Theorem 3.2.5. *Let B be a right A-Galois object. There exists a non-zero invariant functional ψ_B on B.*

Proof. Choose $b \in B$ and put

$$\psi_B^b(b') := \varphi_B(b'_{(0)}b)\psi_A(b'_{(1)}).$$

It is easy to see, using the right invariance property of ψ_A w.r.t. Δ_A, that this functional is invariant. Suppose that ψ_B^b is zero for all $b \in B$. Then by the Galois property of α_B, we have $\varphi_B(b')\psi_A(a) = 0$ for all $b' \in B$ and $a \in A$, which is impossible. So we can choose as ψ_B some non-zero ψ_B^b. □

We prove a uniqueness result concerning the invariant functionals. We can follow the method of Lemma 3.5 and Theorem 3.7 of [93] verbatim.

Proposition 3.2.6. *If ψ_B^1 and ψ_B^2 are two invariant non-zero functionals on an A-Galois object B, then there exists a scalar $\lambda \in k$ such that $\psi_B^1 = \lambda\, \psi_B^2$.*

Proof. First, we show that if ψ_B is a non-zero invariant functional on B, then

$$\{\varphi_B(\,\cdot\, b) \mid b \in B\} = \{\psi_B(\,\cdot\, b) \mid b \in B\}. \tag{3.1}$$

Choose b, b', b'' in B. Then

$$\alpha_B(bb')(b'' \otimes 1) = \sum_i \alpha_B(bw_i)(1 \otimes a_i)$$

for some $w_i \in B, a_i \in A$. If further $b''' \in B, a \in A$, there exist $y_i, z_i \in B$ with

$$\sum_i \alpha_B(by_i)(z_i \otimes 1) = \alpha_B(bb''')(1 \otimes a).$$

If we apply $\psi_B \otimes \varphi_A$ to these expressions we obtain respectively the equalities

$$\begin{aligned} \varphi_B(bb')\psi_B(b'') &= \textstyle\sum_i \psi_B(bw_i)\varphi_A(a_i), \\ \textstyle\sum_i \varphi_B(by_i)\psi_B(z_i) &= \psi_B(bb''')\varphi_A(a). \end{aligned}$$

Choosing either b'' with $\psi_B(b'') = 1$ or a with $\varphi_A(a) = 1$, we get respectively $\subseteq$ and $\supseteq$ of the equality in 3.1.

Suppose now that ψ_B^1 and ψ_B^2 are invariant functionals on B. Choose $b, b_1 \in B$ with $\varphi_B(bb_1) = 1$ and take $b_2 \in B$ with $\psi_B^1(\,\cdot\, b_1) = \psi_B^2(\,\cdot\, b_2)$. Choosing $b' \in B$, applying $\psi_B^i \otimes \varphi_A$ to $(b' \otimes 1)\alpha_B(bb_i)$ and writing this last expression as $\sum_j (1 \otimes a_j)\alpha_B(w_j b_i)$ for certain $w_j \in B, a_j \in A$, we see that $\psi_B^1(b') = \varphi_B(bb_2)\psi_B^2(b')$, proving that all invariant functionals are scalar multiples of each other. □

3.3 The existence of the modular element

Let ψ_B be a non-zero invariant functional on a right A-Galois object B. We prove the existence of a modular element δ_B, relating the functionals φ_B and ψ_B. We first prove an important proposition, which is a kind of strong right invariance formula, familiar from the theory of locally compact quantum groups. (It is easily seen, looking at the proof, that this formula is valid for *any* reduced right coaction that has an invariant functional.)

Proposition 3.3.1. *Let B be a right A-Galois object. For all $b, b' \in B$ we have*

$$S_A((\psi_B \otimes \iota_A)((b \otimes 1)\alpha_B(b'))) = (\psi_B \otimes \iota_A)(\alpha_B(b)(b' \otimes 1)).$$

and

$$S_A((\varphi_B \otimes \iota_A)((b \otimes 1)\alpha_B(b'))) = \delta_A^{-1} \cdot (\varphi_B \otimes \iota_A)(\alpha_B(b)(b' \otimes 1)).$$

Proof. Choose $a \in A$ and $b, b' \in B$. Pick $b_i \in B$ and $a_i \in A$ such that

$$(1 \otimes a)\alpha_B(b') = \sum_i b_i \otimes a_i.$$

Then by the formula for T_{2,α_B}^{-1} given just after Definition 2.5.4, we have

$$b' \otimes S_A(a) = \sum_i \alpha_B(b_i)(1 \otimes S_A(a_i)).$$

If we denote $a' = S_A((\psi_B \otimes \iota_A)((b \otimes 1)\alpha_B(b')))$, then

$$\begin{aligned} a' S_A(a) &= \sum_i \psi_B(bb_i) S_A(a_i) \\ &= \sum_i \psi_B(b_{(0)} b_{i(0)}) b_{(1)} b_{i(1)} S_A(a_i) \\ &= \psi_B(b_{(0)} b') b_{(1)} S_A(a). \end{aligned}$$

Since a was arbitrary, the first formula is proven.

The second formula is proven in completely the same way, only using now that $(\varphi_A \otimes \iota_A)(\Delta_A(a)) = \varphi_A(a)\delta_A$ for $a \in A$.

□

Theorem 3.3.2. *Let B be a right A-Galois object, and ψ_B a non-zero invariant functional. Then ψ_B is faithful, and there exists a unique invertible element $\delta_B \in M(B)$ such that $\varphi_B(b\delta_B) = \psi_B(b)$ for all $b \in B$. Moreover, we have $(\iota \otimes \psi_A)(\alpha_B(b)) = \psi_B(b)\delta_B^{-1}$.*

Proof. We first show that for all $b \in B$:

$$\psi_B(b) = 0 \qquad \Rightarrow \qquad \psi_A(b_{(1)})b_{(0)} = 0. \tag{3.2}$$

We know that $\psi_B(b) = 0$ implies $\psi_B^{b'}(b) = 0$ for all $b' \in B$, i.e.

$$\psi_A(b_{(1)})\varphi_B(b_{(0)}b') = 0$$

for all $b' \in B$. So $\psi_A(b_{(1)})b_{(0)} = 0$ by the faithfulness of φ_B.

Hence if $\psi_B(b'b) = 0$ for all $b' \in B$, then also $\psi_A(b'_{(1)}b_{(1)})b''b'(0)b_{(0)}$ for all $b', b'' \in B$. By the Galois property, $\psi_A(ab_{(1)})b'b_{(0)} = 0$ for all $a \in A$ and $b' \in B$, and then, by the faithfulness of φ_A, the non-degeneracy of B and the faithfulness of α_B, we have $b = 0$. Completely similar, one shows that $\psi_B(bb') = 0$ for all b' implies $b = 0$.

Now from the implication (3.2), it follows that the right hand side is a one-dimensional space, so we can write $\psi_A(b_{(1)})b_{(0)} = \lambda_b\delta'_B$ some number $\lambda_b \in k$ and some multiplier $\delta'_B \in M(B)$, independent of b. Now $b \to \lambda_b$ is easily seen to be a non-zero invariant functional, and replacing ψ_B by this invariant functional (or multiplying δ'_B by some scalar), we obtain $\psi_A(b_{(1)})b_{(0)} = \psi_B(b)\delta'_B$.

Now we show that δ'_B has an inverse δ_B, and that $\varphi_B(b\delta_B) = \psi_B(b)$. Choose $b'' \in B$ with $\psi_B(b') = 1$. Then for $b, b' \in B$, we have, by the previous proposition,

$$\begin{aligned}
\psi_B(bb'\delta'_B) &= \psi_B(bb'b''_{(0)})\varphi_A(S_A(b''_{(1)})) \\
&= \psi_B((bb')_{(0)}b'')\varphi_A((bb')_{(1)}) \\
&= \varphi_B(bb'),
\end{aligned}$$

so by the faithfulness of φ_B, right multiplication with δ'_B is faithful. Since furthermore $\{\varphi_B(\,\cdot\, b') \mid b' \in B\} = \{\psi_B(\,\cdot\, b') \mid b' \in B\}$, we have, by the faithfulness of ψ_B, that for any $b \in B$ there exists $b' \in B$ with $b'\delta'_B = b$, and so right multiplication with δ'_B is surjective.

The non-degeneracy of B easily gives that also left multiplication with δ_B is injective. To show that this operation is also surjective, we use another argument. Take $b \in B$ and $a \in A$ with $\psi_A(a) = 1$. Write $b \otimes a$ as $\sum_i z_{i(0)} w_i \otimes z_{i(1)}$ for certain $z_i, w_i \in B$, and put $b' = \sum_i \psi_B(z_i) w_i$. Then $\delta'_B b' = \sum_i \psi_A(z_{i(1)}) z_{i(0)} w_i = \psi_A(a) b = b$.

Now if l denotes the operation of 'left multiplication with δ_B' and r the operation of 'right multiplication with δ'_B, it is then easy to conclude that $\delta_B := (l^{-1}, (r^{-1})^{\mathrm{op}})$ is a well-defined multiplier of B, and is the inverse of δ'_B.

Then $\psi_B(b\delta'_B) = \varphi_B(b)$ implies

$$\psi_B(b) = \varphi_B(b\delta_B) \qquad \text{for all } b \in B.$$

By the faithfulness of φ_B, this uniquely determines δ_B. □

3.4 The modularity of the invariant functionals

We first prove some identities. The first one is also a variation on the notion of strong (left) invariance.

Proposition 3.4.1. *Let B be a right A-Galois object. For all $b \in B$ and $a \in A$, we have*

i) $\varphi_A(ab_{(1)})b_{(0)} = \varphi_B(a^{[2]}b)a^{[1]}$,

ii) $\varphi_A(b_{(1)}S_A(a))b_{(0)} = \varphi_B(ba^{[1]})a^{[2]}$.

Proof. Using the identities at the end of the first section, the first equation follows from

$$\begin{aligned}\varphi_A(ab_{(1)})b'b_{(0)} &= \varphi_A({a^{[2]}}_{(1)}b_{(1)})b'a^{[1]}{a^{[2]}}_{(0)}b_{(0)} \\ &= \varphi_B(a^{[2]}x)b'a^{[1]},\end{aligned}$$

for all $a \in A$ and $b, b' \in B$. The second follows from

$$\begin{aligned}\varphi_A(b_{(1)}S_A(a))b_{(0)}b' &= \varphi_A(b_{(1)}{a^{[1]}}_{(1)})b_{(0)}{a^{[1]}}_{(0)}a^{[2]}b' \\ &= \varphi_B(ba^{[1]})a^{[2]}b',\end{aligned}$$

for all $a \in A$ and $b, b' \in B$. □

Lemma 3.4.2. *For all $b, b', b'' \in B$ and $a \in A$, we have*

$$\varphi_B(a^{[2]}b)\varphi_B(b'a^{[1]}b'') = \varphi_B(bp^{[1]})\varphi_B(b'p^{[2]}b''),$$

where $p = (S_A^{-1}\sigma_A)(a)$.

Proof. Using the identities of the previous lemma, we get

$$\begin{aligned} \varphi_B(bp^{[1]})\varphi_B(b'p^{[2]}b'') &= \varphi_A(b_{(1)}\sigma_A(a))\varphi_B(b'b_{(0)}b'') \\ &= \varphi_A(ab_{(1)})\varphi_B(b'b_{(0)}b'') \\ &= \varphi_B(a^{[2]}b)\varphi_B(b'a^{[1]}b''). \end{aligned}$$

□

We show now that φ_B possesses a modular automorphism.

Theorem 3.4.3. *Let B be a right A-Galois object. There exists an automorphism σ_B of B such that*

$$\varphi_B(b\sigma_B(b')) = \varphi_B(b'b) \qquad \textit{for all } b, b' \in B.$$

Furthermore, $\varphi_B \circ \sigma_B = \varphi_B$.

Proof. Choose $b \in B$. We can then write

$$b = \sum_i \varphi_B(b_i'a_i^{[1]}b_i'')a_i^{[2]}$$

for certain $b_i', b_i'' \in B$ and $a_i \in A$, since $B^2 = B$ and the map $b \otimes a \to ba^{[1]} \otimes a^{[2]}$ is bijective. Define

$$b''' = \sum_i \varphi_B(b_i'p_i^{[2]}b_i'')p_i^{[1]}$$

with $p_i = (S_A^{-1}\sigma_A)(a_i)$. Then the previous lemma shows that $\varphi_B(b'b''') = \varphi_B(bb')$ for all $b' \in B$.

It is clear that b''' is uniquely determined by this property, by faithfulness of φ_B, so we can denote $b''' = \sigma_B(b)$. An easy argument shows that σ_B is a homomorphism. Again by faithfulness of φ_B, it is faithful. To see that it is surjective, simply reverse the argument in the first paragraph to obtain that for any b, there exists $\sigma_B^{-1}(b)$ such that $\varphi_B(\sigma_B^{-1}(b)b') = \varphi_B(b'b)$ for all $b' \in B$. Then $\sigma_B(\sigma_B - 1(b)) = b$. Hence σ_B is an automorphism. It will leave φ_B invariant because $B^2 = B$. □

Remarks: 1. As for algebraic quantum groups, the concrete way in which σ_B is constructed is not so important. What *is* important is its modular property, which makes up for the fact that φ_B does not have to be tracial.
2. It is easily seen that the defining property of σ_B also holds for multipliers: if $b \in B$ and $m \in M(B)$, then $\varphi_B(b\sigma_B(m)) = \varphi_B(mb)$, and $\varphi(m\sigma_B(b)) = \varphi(bm)$.

Corollary 3.4.4. *Let ψ_B be an invariant functional on a Galois object B. Then the functional ψ_B is modular with modular automorphism*

$$\sigma_B(b) = \delta_B \cdot \sigma_B(b) \cdot \delta_B^{-1}.$$

3.5 Formulas

In this section and the next, we collect some formulas. They strongly resemble the formulas which hold in algebraic quantum groups, and also their proofs are mostly straightforward adaptations.

Proposition 3.5.1. *Let B be a right A-Galois object. For all $a \in A$, we have*

i) $\alpha_B \circ \sigma_B = (\sigma_B \otimes S_A^{-2}) \circ \alpha_B$,

ii) $((S_A^{-1}\sigma_A)(a))^{[1]} \otimes ((S_A^{-1}\sigma_A)(a))^{[2]} = \sigma_B(a^{[2]}) \otimes a^{[1]}$.

Proof. Choose $b, b' \in B$ and $a \in A$. Then using Proposition 3.3.1 twice, we get

$$\begin{aligned}(\psi_B \otimes \varphi_A)((b' \otimes a)\alpha_B(\sigma_B(b))) &= \psi_B(b'_{(0)}\sigma_B(b))\varphi_A(aS_A^{-1}(b'_{(1)}))\\ &= \psi_B(bb'_{(0)})\varphi_A(aS_A^{-1}(b'_{(1)}))\\ &= \psi_B(b_{(0)}b')\varphi_A(aS_A^{-2}(b_{(1)}))\\ &= \psi_B(b'\sigma_B(b_{(0)}))\varphi_A(aS_A^{-2}(b_{(1)})),\end{aligned}$$

applying Proposition 3.3.1 twice. As φ_A and ψ_B are faithful, the first identity follows. The second formula was essentially proven in Lemma 3.4.2. □

Corollary 3.5.2. *Let B be a right A-Galois object. Then the maps*

$$\begin{aligned}&b \otimes a \to \tilde{\beta}_A(a)(b \otimes 1),\\ &b \otimes a \to (1 \otimes b)\tilde{\beta}_A(a)\end{aligned}$$

are bijections from $B \odot A$ to $B \odot B$

Proof. This follows from the second formula of the previous proposition. □

Note that this fact is not at all clear at first sight. In particular, it allows us to view $\tilde{\beta}_A$ as a map $A \to M_{1;2}(B \odot B)$. Denote then by C the algebra B^{op}, and by S_C the canonical map $C \to B$ sending b^{op} to b for $b \in B$. We can then give meaning to $\beta_A := (S_C^{-1} \otimes \iota)\tilde{\beta}_A$ as a map $A \to M_{1;2}(C \odot B)$. Now as is the case for Galois objects over Hopf algebras, the map β_A will then be a (u.u.e.) homomorphism. The argument for this is simple: choose $b \in B$ and $a, a' \in A$, and write $ba'^{[1]} \otimes a'^{[2]} = \sum_i z_i \otimes w_i$ for certain $z_i, w_i \in B$. Then $\sum_i (z_i \otimes a)\alpha_B(w_i) = b \otimes aa'$. Applying G^{-1}, we obtain $\sum_i z_i a^{[1]} \otimes a^{[2]} w_i = a(aa')^{[1]} \otimes (aa')^{[2]}$, so $ba'^{[1]}a^{[1]} \otimes a^{[2]}a'^{[2]} = b(aa')^{[1]} \otimes (aa')^{[2]}$. This proves that β_A is a homomorphism.

We can then also construct a *Miyashita-Ulbrich* action of the algebraic quantum group A on a right Galois object B for it. This is a right A-module structure on B, defined as

$$b \cdot a := a^{[1]} b a^{[2]}.$$

One can then show that it satisfies a certain property with respect to the coaction structure, making it a *Yetter-Drinfel'd module*, but we will not go into this here.

Definition 3.5.3. *Let B be a right Galois object for an algebraic quantum group A. We call the homomorphism $\beta_A : A \to M(C \odot B)$ constructed above the* external comultiplication *on A.*

In the following, we will always use the symbol C to denote B^{op}. We will also use a Sweedler notation for the map β_A in the following way: $\beta_A(a) = a_{[1]} \otimes a_{[2]}$ for $a \in A$.

The following proposition collects some formulas concerning the modular elements.

Proposition 3.5.4. *The following identities hold:*

iii) $\alpha_B(\delta_B) = \delta_B \otimes \delta_A$,

iv) $\beta_A(\delta_A) = \delta_C \otimes \delta_B$, *where* $\delta_C = (\delta_B^{-1})^{op} \in C$,

v) $\sigma_B(\delta_B) = \nu_A^{-1} \delta_B$.

Proof. For $b, b' \in B$, we have, using the second formula in Proposition 3.3.1, that

$$\begin{aligned}\varphi_B(b(b'\delta_B)_{(0)})(b'\delta_B)_{(1)} &= \varphi_B(b_{(0)}b'\delta_B)S_A^{-1}(b_{(1)})\delta_A \\ &= \psi_B(b_{(0)}b')S_A^{-1}(b_{(1)})\delta_A \\ &= \psi_B(bb'_{(0)})b'_{(1)}\delta_A \\ &= \varphi_B(bb'_{(0)}\delta_B)b'_{(1)}\delta_A.\end{aligned}$$

By faithfulness of φ_B we have $\alpha_B(b'\delta_B) = \alpha_B(b')(\delta_B \otimes \delta_A)$, hence $\alpha_B(\delta_B) = \delta_B \otimes \delta_A$ by definition of α_B on $M(B)$.

For the second formula, we have to prove that

$$b(a\delta_A)^{[1]} \otimes (a\delta_A)^{[2]} = b\delta_B^{-1}a^{[1]} \otimes a^{[2]}\delta_B$$

for all $a \in A$ and $b \in B$. This follows immediately by applying G and using the previous formula.

As for the final formula, we have for any $b \in B$ that

$$\begin{aligned}\varphi_B(\delta_B b) &= \varphi_A(\delta_A b_{(1)})\delta_B b_{(0)} \\ &= \nu_A^{-1}\varphi_A(b_{(1)}\delta_A)\delta_B(b_{(0)}\delta_B)\delta_B^{-1} \\ &= \nu_A^{-1}\varphi_A((b\delta_B)_{(1)})\delta_B(b\delta_B)_{(0)}\delta_B^{-1} \\ &= \nu_A^{-1}\varphi_B(b\delta_B),\end{aligned}$$

which means exactly that $\sigma_B(\delta_B) = \nu_A^{-1}\delta_B$. □

Corollary 3.5.5. *If B is a right A-Galois object, and if φ'_B is a δ_A-invariant functional, then there exists $\lambda \in k$ with $\varphi'_B = \lambda\,\varphi_B$.*

Proof. This follows immediately by the uniqueness of an invariant functional and the fact that $\varphi'_B(\,\cdot\,\delta_B)$ is invariant. □

3.6 The square of an antipode

Let B be a right A-Galois object. There is a natural unital left $\widehat{A}$-module algebra structure on B defined by

$$\omega \cdot b := (\iota_B \otimes \omega)\alpha_B(b)$$

for $b \in B$ and $\omega \in \widehat{A}$. The unitality, together with the existence of local units in $\widehat{A}$, allows us to extend the $\widehat{A}$-module structure on B to a left $M(\widehat{A})$-module structure on B: if $m \in M(\widehat{A})$, we let it act on an element $b = \sum_i \omega_i \cdot b_i$ in B as

$$m(\sum_i \omega_i \cdot b_i) := \sum_i ((m \cdot \omega_i) \cdot b_i).$$

This is independent of the chosen representation of b, since if $\sum_i \omega_i \cdot b_i = 0$, we can take $\omega \in \widehat{A}$ with $\omega \cdot \omega_i = \omega_i$ for all i, and then

$$\begin{aligned} \sum_i (m\omega_i) \cdot b_i &= \sum_i (m\omega\omega_i) \cdot b_i) \\ &= \sum_i (m\omega) \cdot (\omega_i \cdot b_i) \\ &= 0. \end{aligned}$$

It is further easy to check then that for $\omega \in M(\widehat{A})$ and $b, b' \in B$, we have

$$b' \cdot (\omega \cdot b) = (\iota_B \otimes \omega)((b' \otimes 1)\alpha_B(b)),$$

and

$$(\omega \cdot b) \cdot b' = (\iota_B \otimes \omega)(\alpha_B(b)(b' \otimes 1)),$$

where we have interpreted $M(\widehat{A}) \subseteq A^*$.

Consider the map

$$S_B^2 : B \to B : b \to \sigma_B(\delta_{\widehat{A}} \cdot b),$$

where $\delta_{\widehat{A}}$ is the modular element of the dual $(\widehat{A}, \Delta_{\widehat{A}})$, and where the 'square' is just formal (i.e., does not really denote the square of something).

Proposition 3.6.1. *Let B be a right A-Galois object. Then S_B^2 is a bijective homomorphism.*

Proof. The bijectivity is clear, since

$$S_B^{-2} : B \to B : b \to \delta_{\widehat{A}}^{-1} \cdot (\sigma_B^{-1}(b))$$

is an inverse for S_B^2. As for the fact that S_B^2 is a homomorphism, it is sufficient to check that

$$\delta_{\widehat{A}} \cdot (bb') = (\delta_{\widehat{A}} \cdot b) \cdot (\delta_{\widehat{A}} \cdot b').$$

But since $\delta_{\widehat{A}} = \varepsilon_A \circ \sigma_A^{-1}$ is a homomorphism $\widehat{A} \to k$, we have that

$$\begin{aligned} \delta_{\widehat{A}}(b \cdot b') &= (\iota_B \otimes \delta_{\widehat{A}})(\alpha_B(bb')) \\ &= (\iota_B \otimes \delta_{\widehat{A}})(\alpha_B(b)\alpha_B(b')) \\ &= (\iota_B \otimes \delta_{\widehat{A}})(\alpha_B(b))(\iota_B \otimes \delta_{\widehat{A}})(\alpha_B(b')) \\ &= (\delta_{\widehat{A}} \cdot b) \cdot (\delta_{\widehat{A}} \cdot b'). \end{aligned}$$

□

This map S_B^2 plays the rôle of 'the square of the antipode' for B, hence the notation. Indeed: in case $B = A$ and $\alpha_B = \Delta_A$, then S_B^2 is exactly S_A^2, using Proposition 2.4.3 and the commutation relations in Definition-Proposition 2.4.2. Some more reasons to consider this as an antipode squared will be provided further on.

We can use S_B^2 to complete our set of formulas.

Proposition 3.6.2. *The following identities hold:*

vi) $\alpha_B \circ \sigma_B = (S_B^2 \otimes \sigma_A) \circ \alpha_B$,

vii) $\alpha_B \circ S_B^2 = (S_B^2 \otimes S_A^2) \circ \alpha_B$,

viii) $\alpha_B \circ S_B^2 = (\sigma_B \otimes \sigma_A^{-1}) \circ \alpha_B$,

ix) $\sigma_B \circ S_B^2 = S_B^2 \circ \sigma_B$,

x) $S_B^2(\delta_B) = \delta_B$,

xi) $\varphi_B(S_B^2(b)) = \varphi_B(\delta_B^{-1} b \delta_B) = \nu_A \varphi_B(b)$ *for* $b \in B$.

Proof. Take $b, b' \in B$ and $a \in A$. Then, using again Proposition 2.4.3, the second identity of Proposition 3.3.1 and the commutation relations in Definition-Proposition 2.4.2, we find

$$\begin{aligned} \varphi_B(b' S_B^2(b_{(0)}))\varphi_A(a\sigma_A(b_{(1)})) &= \varphi_B((\delta_{\widehat{A}} \cdot b_{(0)})b')\varphi_A(b_{(1)}a) \\ &= \varphi_B(b_{(0)}b')\varepsilon_A(\sigma_A^{-1}(b_{(1)}))\varphi_A(b_{(2)}a) \\ &= \varphi_B(b_{(0)}b')\varphi_A(S_A^{-2}(\sigma_A^{-1}(b_{(1)}))a) \\ &= \varphi_B(bb'_{(0)})\psi_A(aS_A^{-1}(b'_{(1)})) \\ &= \varphi_B(b'_{(0)}\sigma_B(b))\psi_A(aS_A^{-1}(b'_{(1)})) \\ &= \varphi_B(b'\sigma_B(b)_{(0)})\varphi_A(a\sigma_B(b)_{(1)}). \end{aligned}$$

This proves the equality in *vi*). The equality in *vii*) then follows by the previous one, using that $\alpha_B(\delta_{\hat{A}}(b)) = b_{(0)} \otimes (\delta_{\hat{A}} \cdot b_{(1)})$ as multipliers.

Further,

$$\begin{aligned}\varphi_B(b' S_B^2(b_{(0)}))\psi_A(S_A^2(b_{(1)})a) &= \varphi_B(b'\sigma_B(b_{(0)}))\delta_{\hat{A}}(b_{(1)})\psi_A(S_A^2(b_{(2)})a)\\ &= \varphi_B(b'\sigma_B(b_{(0)}))\varepsilon_A(\sigma_A^{-1}(b_{(1)}))\psi_A(S_A^2(b_{(2)})a)\\ &= \varphi_B(b'\sigma_B(b_{(0)}))\psi_A(\sigma_A^{-1}(b_{(1)})a),\end{aligned}$$

which together with *vii*) proves *viii*).

By *vi*), it follows that also $S_B^2(b) = \delta_{\hat{A}} \cdot (\sigma_B(b))$, whence the commutation in *ix*). As for *x*) we have $S_B^2(\delta_B) = \sigma_B(\delta_B)\varepsilon_A(\sigma_A^{-1}(\delta_A))$, which equals δ_B by the formula *v*). The same formula *v*) also shows immediately the validity of *xi*). This concludes the proof. □

Recall that we already constructed a map $S_C : C \to B$, which was just the canonical linear map $B^{\mathrm{op}} \to B$.

Definition 3.6.3. *Let B be a right A-Galois object. We call the map*

$$S_C : C \to B : b^{op} \to b$$

the antipode on C. *We call the map*

$$S_B : B \to C : b \to (S_B^2(b))^{op}$$

the antipode on B.

Then indeed, $S_C \circ S_B = S_B^2$, so that S_B^2 can be considered to be 'the square of an antipode'!... If the reader feels cheated at this point, we urge him to read on.

For example, the following formulas should give a more direct connection with the defining property of an antipode. We will also write $S_C^2(b^{\mathrm{op}}) = (S_B^2(b))^{\mathrm{op}}$ for $b \in B$, and continue to use the Sweedler notation for β_A, introduced after Definition 3.5.3.

Proposition 3.6.4. *Let B be a right A-Galois object. For all $b \in B, c \in C$ and $a \in A$, we have*

xiii) $S_A(a)_{[1]} \otimes S_A(a)_{[2]} = S_B(a_{[2]}) \otimes S_C(a_{[1]})$, *xiv)* $cS_B(b_{(0)})b_{(1)[1]} \otimes b_{(1)[2]} = c \otimes b$,

xv) $ca_{[1]}S_B(a_{[2]}) = \varepsilon_A(a)c$,

xvi) $\beta_A \circ S_A^2 = (S_C^2 \otimes S_B^2) \circ \beta_A$,

xvii) $\beta_A \circ \sigma_A = (S_C^2 \otimes \sigma_B) \circ \beta_A$

Remark: Note that $S_B \otimes S_C$ is well-defined on $M(C \otimes B)$, so the first identity makes sense.

Proof. Applying $(\iota_B \otimes \varphi_B(\,\cdot\, b))$ to $S_B^2(a^{[2]}) \otimes a^{[1]}$ and using formula *vi*), we get

$$\begin{aligned}
\varphi_B(a^{[1]}b)S_B^2(a^{[2]}) &= \varphi_B(\sigma_B^{-1}(b)a^{[1]})S_B^2(a^{[2]}) \\
&= \varphi_A(\sigma_B^{-1}(b)_{(1)}S_A(a))S_B^2(\sigma_B^{-1}(b)_{(0)}) \\
&= \varphi_A(\sigma_A^{-1}(b_{(1)})S_A(a))b_{(0)} \\
&= \varphi_A(S_A(a)b_{(1)})b_{(0)} \\
&= \varphi_B(S_A(a)^{[2]}b)S_A(a)^{[1]},
\end{aligned}$$

so that $S_A(a)^{[1]} \otimes S_A(a)^{[2]} = S_B^2(a^{[2]}) \otimes a^{[1]}$. This is easily seen to be equivalent with the first formula.

As for the second formula, we have to show, applying $\varphi_B(\,\cdot\, b')$ to the second leg and writing $c = (b'')^{\mathrm{op}}$, that for all $b' \in B$ we have

$$\varphi_B(b_{(1)}^{[2]}b')b_{(1)}^{[1]}S_B^2(b_{(0)})b'' = \varphi_B(bb')b''.$$

This reduces, by Proposition 3.4.1.(i), to proving that

$$\varphi_A(b_{(1)}b'_{(1)})b'_{(0)}S_B^2(b_{(0)})b'' = \varphi_B(bb')b''.$$

This follows again by formula *vi*) and the defining property of φ_B.

The last formulae are a direct consequence of the first (using Proposition 3.5.1. *ii*) for the last one). □

Note that the second identity in the last proposition shows that

$$C \odot B \to C \odot A : c \otimes b \to cS_B(b_{(0)}) \otimes b_{(1)}$$

is the inverse of the map

$$C \odot A \to C \odot B : c \otimes a \to ca_{[1]} \otimes a_{[2]},$$

which correspond to the exact same formula for a (multiplier) Hopf algebra if we replace C and B by A, S_B by S_A and β_A by the comultiplication map. More directly, we also have that $S_C(a_{[1]})a_{[2]} = \varepsilon_A(a)1 = a_{[1]}S_B(a_{[2]})$ (where the unit in the middle is really in different algebras for the left and right expression).

However, we want to give a little warning at this point, as the situation could get a bit confusing when we consider $(B, \alpha_B) = (A, \Delta_A)$ (which is evidently a right Galois object for A). For then we have an antipode S_A for the *algebraic quantum group* (A, Δ_A), which will be an anti-isomorphism $A \to A$, but we also have an antipode S_B for the *Galois object* A, which will be an anti-isomorphism $A \to A^{\mathrm{op}}$. In some sense, for an algebraic quantum group the antipode contains extra information, which is not present in its square. But for a Galois object, the antipode is really just a formal construction *using* its antipode squared.

We want to remark that the notion of an 'antipode squared' on a Galois object for a Hopf algebra was considered more or less in [43], but in a different set-up. Also, the antipode squared there was a part of the axiom system. The connection with Galois objects and the redundancy of having this 'antipode squared' in the axiom system, was established in [75]. The notion of an antipode for a Galois object was considered explicitly first in [8] (although it seems to have been implicit in earlier work by Schauenburg). As a final remark, note that we can easily get into Grünspans framework of quantum torsors, by means of the quantum torsor map

$$(\iota_B \otimes \beta_A)\alpha_B : B \to M(B \otimes B^{\mathrm{op}} \otimes B).$$

However, we have not developed an independent theory for such 'algebraic quantum torsors' (which seems very plausible to exist).

3.7 The inverse Galois object

In the discussion up to now, we have worked exclusively with right Galois objects. Of course, there is also the notion of a *left* Galois object, and all results obtained for right Galois objects have their counterparts in the left setting. But the correspondence between right and left Galois objects is more than a formal one: there is a natural one-to-one correspondence between right A-Galois objects and left A-Galois objects. For given a right A-Galois object, we can turn $C = B^{\mathrm{op}}$ into a left A-Galois object in a straightforward fashion.

Definition-Proposition 3.7.1. *Let B be a right A-Galois object The map*

$$\gamma_C : C \to M(A \odot C) : c \to (S_A^{-1} \otimes S_C^{-1})\alpha_B^{op}(S_C(c))$$

makes C into a left A-Galois object, which we then call the inverse Galois object *(of B).*

Proof. Since $(S_A^{-1} \otimes S_C^{-1})$ is an anti-isomorphism $A \odot B \to A \odot C$, it is clear that we can extend it to an anti-isomorphism $M(A \odot B) \to M(A \odot C)$, so that $\gamma_C(c)$ for $c \in C$ is meaningful as an element of $M(A \odot C)$. It is also easy to check that γ_C gives us a reduced coaction. Since the Galois map is given by the formula

$$b^{\mathrm{op}} \odot b'^{\mathrm{op}} \to S_A^{-1}(b_{(1)}) \otimes (b' b_{(0)})^{\mathrm{op}},$$

it is bijective, by the remark following Definition 2.5.4.

□

It is easy to see that $\varphi_C = \psi_B \circ S_B^{-1}$ and $\psi_C = \varphi_B \circ S_C$ provide resp. an invariant and δ_A^{-1}-invariant functional, using some of the identities established earlier on. We also state (without proof) that the modular element δ_C connecting these two functionals is $\delta_C = (\delta_B^{-1})^{\mathrm{op}}$, and that the modular automorphisms σ_C and σ_C of resp. φ_C and ψ_C are given by $\sigma_C(b^{\mathrm{op}}) = (\sigma_B^{-1}(b))^{\mathrm{op}}$ and $\sigma_C(b^{\mathrm{op}}) = (\sigma_B^{-1}(b))^{\mathrm{op}}$. The antipode $S_C : C \to C^{\mathrm{op}} = B$ for C coincides with the one already introduced.

We also have the following coassociativity properties:

Proposition 3.7.2. *Let B be a right A-Galois object. Then for all $a \in A$, we have*

$$(\iota_C \otimes \alpha_B)(\beta_A(a)) = (\beta_A \otimes \iota_A)(\Delta_A(a))$$

and

$$(\gamma_C \otimes \iota_B)(\beta_A(a)) = (\iota_A \otimes \beta_A)(\Delta_A(a)).$$

Proof. Note first that the maps $\beta_A \otimes \iota_A$ and $\iota_A \otimes \beta_A$ are u.u.e., so the statement makes sense.

Now if $a, a' \in A$ and $b \in B$, then we compute, using Proposition 3.1.2 and the identity following it, that

$$\begin{aligned}
&(G \otimes \iota_A)(ba_{(1)}^{[1]} \otimes a_{(1)}^{[2]} \otimes a_{(2)}a') \\
&= b \otimes a_{(1)} \otimes a_{(2)}a' \\
&= (\iota_B \otimes \Delta_A)(b \otimes a)(1_B \otimes 1_A \otimes a') \\
&= (\iota_B \otimes \Delta_A)(ba^{[1]}a^{[2]}_{(0)} \otimes a^{[2]}_{(1)})(1_B \otimes 1_A \otimes a') \\
&= ba^{[1]}a^{[2]}_{(0)} \otimes a^{[2]}_{(1)} \otimes a^{[2]}_{(2)}a' \\
&= (G \otimes \iota_A)(ba^{[1]} \otimes a^{[2]}_{(0)} \otimes a^{[2]}_{(1)})a',
\end{aligned}$$

which proves the first identity in the lemma.

As for the second statement, this reduces to proving that

$$S_A^{-1}(a^{[1]}_{(1)}) \otimes a^{[1]}_{(0)} \otimes a^{[2]} = a_{(1)} \otimes a_{(2)}^{[1]} \otimes a_{(2)}^{[2]}.$$

But again using Proposition 3.1.2 and the identity which follows it, we have that

$$\begin{aligned}
&(\iota_A \otimes H)(a'a_{(1)} \otimes a_{(2)}^{[1]} \otimes a_{(2)}^{[2]}b) \\
&= a'a_{(1)} \otimes b \otimes S_A(a_{(2)}) \\
&= a'S_A^{-1}(S_A(a)_{(2)}) \otimes b \otimes S_A(a)_{(1)} \\
&= (\iota_A \otimes H)(a'S_A^{-1}(a^{[1]}_{(1)}) \otimes a^{[1]}_{(0)} \otimes a^{[2]}b).
\end{aligned}$$

□

3.8 Galois objects of compact or discrete type

Definition 3.8.1. *A non-degenerate algebra B is called* of compact type *if B has a unit. It is called* of discrete type *if every subspace of the form bB or Bb, with $b \in B$, is finite dimensional.*

Remark: This terminology is not standard, and we use it solely in this subsection.

Theorem 3.8.2. *Let B be a right Galois object for an algebraic quantum group A. Then the algebra B is of compact type iff A is an algebraic quantum group of compact type. The algebra B is of discrete type iff A is an algebraic quantum group of discrete type.*

We recall from [93] that an algebraic quantum group A is of compact type if A has a *unital* underlying algebra (i.e. is a Hopf algebra with integrals), and that A is of discrete type if there exists a *cointegral* $h \in A$, i.e. an element satisfying $ah = \varepsilon_A(a)h$ for all $a \in A$. We also note that an algebraic quantum group is of compact type iff its dual is of discrete type.

Proof. If A is compact, then $\alpha_B(b) \in B \otimes A$ for any $b \in B$. Choosing $b \in B$ with $\varphi_B(b) = 1$, we have that $(\iota_B \otimes \varphi_A)\alpha_B(b) \in B$ is a unit of B.

If B is compact, choose $a_i \in A$ and $b_i \in B$ such that

$$1_C \otimes 1_B = \sum_i \beta_A(a_i)(1 \otimes b_i).$$

Taking $a \in A$ and multiplying the above equality to the left with $\beta_A(a)$, we get

$$\beta_A(a)(1_C \otimes 1_B) = \sum_i \beta_A(aa_i)(1 \otimes b_i),$$

hence, by the bijectivity of the maps in Proposition 3.1.2, we conclude $a \otimes 1_B = \sum_i aa_i \otimes b_i$. Applying an arbitrary $\omega \in B^*$ with value 1 in 1_B to the second leg, we see that A has a right unit. Similarly, one constructs a left unit. So A is unital.

Now suppose that A is an algebraic quantum group of discrete type. Choose a non-zero left cointegral $h \in A$, so $ah = \varepsilon_A(a)h$ for all $a \in A$. We can scale h so that $\varphi_A(h) = 1$. Then for all $b, b' \in B$, we have

$$\begin{aligned} \varphi_B(b(S_A^{-1}(h))^{[1]})(S_A^{-1}(h))^{[2]}b' &= \varphi_A(b_{(1)}h)b_{(0)}b' \\ &= bb' \end{aligned}$$

by Proposition 3.4.1 *ii*). Hence if $S_A^{-1}(h))^{[1]}) \otimes (S_A^{-1}(h))^{[2]}b' = \sum_i p_i \otimes q_i$, we see that for any $b \in B$, the element bb' lies in the linear span of the q_i. This shows that Bb' is finite dimensional. Also $b'B$ is finite dimensional, by a similar reasoning.

Conversely, suppose that B is an algebra of discrete type. Take $a \in A$ and $b \neq 0$ fixed in B. Write $ba^{[1]} \otimes a^{[2]}$ as $\sum_i w_i \otimes z_i$, and choose $b' \in B$ such that $w_i b' = w_i$ for all i (Corollary 3.2.3). Then

$$\begin{aligned}\dim(Aa) &= \dim\{b \otimes a'a \mid a' \in A\} \\ &= \dim\{\sum_i w_i b' a'^{[1]} \otimes a'^{[2]} z_i \mid a' \in A\} \\ &\leqslant \dim \operatorname{span}\{\sum_i w_i b'' \otimes b''' z_i \mid b'', b''' \in B\} \\ &< \infty.\end{aligned}$$

We show that this is sufficient to conclude that A is an algebraic quantum group of discrete type.

First, applying S_A, we see that also all aA are finite dimensional. Choose $a \in A$ with $\varepsilon_A(a) = 1$. Write $I = AaA$, which is a finite-dimensional ideal. Because φ_A is faithful, we can choose some $\omega = \varphi_A(\cdot a') \in \widehat{A}$ such that $\omega_{|I} = (\varepsilon_A)_{|I}$. Take $e \in A$ with $ae = a$. Then for all $a'' \in A$, we have

$$\begin{aligned}\varphi_A(a''aa') &= \varphi_A(a''aea') \\ &= \omega_{|I}(a''ae) \\ &= \varepsilon_A(a'').\end{aligned}$$

Hence $\varepsilon_A \in \widehat{A}$, and A is an algebraic quantum group of discrete type. □

Note that the proof above shows that the terminology we used is consistent: an algebraic quantum group is of discrete type in the sense of [93] iff its underlying algebra is of discrete type as defined in Definition 3.8.1. Also note that if $k = \mathbb{C}$ and B is a *-algebra, the condition 'B is of discrete type' is equivalent with B being a direct sum of finite-dimensional matrix algebras.

Proposition 3.8.3. *If A is an algebraic quantum group of discrete type, and B a right Galois object for A, then B is a Frobenius algebra in the sense of [98]: there exists a left B-module isomorphism $L : BB^* \to B$, where B^* is the dual space of B.*

Here BB^* denotes functionals of the form $b\cdot\omega = \omega(\,\cdot\, b)$ for $\omega \in B^*$ and $b \in B$.

Proof. Let p be the right cointegral of A, so that $pa = \varepsilon_A(a)p$ for all $a \in A$. We assume p normalized, so that $\varphi_A(p) = 1$. We show then that

$$(b \otimes 1)\tilde{\beta}_A(p) = \tilde{\beta}_A(p)(1 \otimes b)$$

for all $b \in B$.

Take $b, b' \in B$ and apply $(\iota_B \otimes \varphi_B(\,\cdot\, b'))$ to $\tilde{\beta}_A(p)(1 \otimes b)$. Then we find, using the first identity in Proposition 3.4.1

$$\begin{aligned}
(\iota_B \otimes \varphi_B)(\tilde{\beta}_A(p)(1 \otimes bb')) &= \varphi_B(p^{[2]}bb')p^{[1]} \\
&= \varphi_A(pb_{(1)}b'_{(1)})b_{(0)}b'_{(0)} \\
&= \varepsilon_A(b_{(1)}b'_{(1)})b_{(0)}b'_{(0)} \\
&= bb' \\
&= \varphi_A(pb'_{(1)})bb'_{(0)} \\
&= \varphi_B(p^{[2]}b')bp^{[1]} \\
&= (\iota_B \otimes \varphi_B)((b \otimes 1)\tilde{\beta}_A(p)(1 \otimes b')).
\end{aligned}$$

As φ_B is faithful, this implies $(b \otimes 1)\tilde{\beta}_A(p) = \tilde{\beta}_A(p)(1 \otimes b)$ for all $b \in B$.

Consider then

$$\phi_1 : BB^* \to B : \omega \to (\iota \otimes \omega)(\tilde{\beta}_A(p)),$$
$$\phi_2 : B \to BB^* : b \to \varphi_B(\cdot b).$$

Then ϕ_1 and ϕ_2 are seen to be B-module morphisms, using the above identity. Moreover, they are each others inverse: choose $b \in B$ and $\omega \in BB^*$, then, by the second identity in Proposition 3.4.1,

$$\begin{aligned}
\varphi_B(b \cdot (\iota_B \otimes \omega)(\tilde{\beta}_A(p))) &= \varphi_B(bp^{[1]})\omega(p^{[2]}) \\
&= \varphi_A(b_{(1)}S_A(p))\omega(b_{(0)}) \\
&= \varphi_A(S_A(p))\omega(b) \\
&= \varphi_A(p\delta_A)\omega(b) \\
&= \omega(b),
\end{aligned}$$

showing that $\phi_2 \circ \phi_1$ is the identity. The fact that $\phi_1 \circ \phi_2$ is the identity follows from $\varphi_B(p^{[2]}b)p^{[1]} = b$ for all $b \in B$. □

3.9 *-Structures on Galois objects

We now look at the case $k = \mathbb{C}$.

Definition 3.9.1. *Let B be a completely positive *-algebra, and let A be a *-algebraic quantum group. If $\alpha_B : B \to M(B \odot A)$ is a coaction making (B, α_B) into a right Galois object for A (neglecting the *-structure), we call (B, α_B) a right* *-Galois object *if α_B is *-preserving.*

Proposition 3.9.2. *Let B be a right *-Galois object for a *-algebraic quantum group A. Then the functional $\varphi_B = (\iota_B \otimes \varphi_A)\alpha_B$ is positive.*

Proof. We have to show that $\varphi_B(b^*b) \geqslant 0$ for all $b \in B$.

First remark that φ_B is hermitian: if $b, b' \in B$, we have

$$\begin{aligned}
\varphi_B(b^*)b' &= \varphi_A((b^*)_{(1)})(b^*)_{(0)}b' \\
&= \varphi_A((b_{(1)})^*)(b_{(0)})^*b' \\
&= \overline{\varphi_A(b_{(1)})}(b'^*b_{(0)})^* \\
&= (\varphi_A(b_{(1)})b'^*b_{(0)})^* \\
&= \overline{\varphi_B(b)}b'^*,
\end{aligned}$$

hence $\varphi_B(b^*) = \overline{\varphi_B(b)}$.

Now take non-zero $b, b' \in B$, and write $\alpha_B(b)(b' \otimes 1) = \sum w_i \otimes p_i$ for certain $w_i \in B$ and $p_i \in A$. Then

$$\begin{aligned}
\varphi_B(b^*b)b'^*b' &= (\iota_B \otimes \varphi_A)((\alpha_B(b)(b' \otimes 1))^*(\alpha_B(b)(b' \otimes 1))) \\
&= \sum_{i,j} \varphi_A(p_j^*p_i)w_j^*w_i.
\end{aligned}$$

By positivity of φ_A, the matrix $(\varphi_A(p_j^*p_i))_{i,j}$ will be positive, so that we can write $\varphi_B(b^*b)b'^*b' = \sum_i z_i^*z_i$ for certain $z_i \in B$. Then $\varphi_B(b^*b)$, which is a real number, must necessarily be positive, or else we would violate the complete positivity of B.

□

There is a nice formula relating β_A and the *-operation, but for this, we have to choose the good *-operation on $C = B^{\mathrm{op}}$: we define $(b^{\mathrm{op}})^* := S_B^2(b^*)^{\mathrm{op}}$. Then C is again a *-algebra: the only thing which may not be clear at first sight, is if the *-operation is involutive, that is, if $S_B^2(b^*) = S_B^{-2}(b)^*$. For this, note first that $\sigma_B(b^*) = \sigma_B^{-1}(b)^*$: one verifies this by checking that for all $b, b' \in B$, we have $\varphi_B(\sigma_B(b^*)^*b') = \varphi_B(b'b)$, using that φ_B is hermitian. Then note that $(\delta_{\widehat{A}} \cdot b)^* = \delta_{\widehat{A}}^{-1} \cdot b^*$: for this, observe that

$$\begin{aligned}
\delta_{\widehat{A}}(a^*) &= \varepsilon_A(\sigma_A^{-1}(a^*)) \\
&= \varepsilon_A(\sigma_A(a)^*) \\
&= \overline{\delta_{\widehat{A}}^{-1}(a)}.
\end{aligned}$$

Then with this *-operation, we have the expected formulas

$$S_C((c)^*) = (S_B^{-1}(c))^*$$

and

$$S_B(b^*) = (S_C^{-1}(b))^*$$

for $c \in C$ and $b \in B$.

Proposition 3.9.3. *For all $a \in A$, we have*

$$\beta_A(a)^* = \beta_A(a^*).$$

Proof. For any $b \in B$, $a \in A$, we have

$$\varphi_A(ab_{(1)})b_{(0)} = \varphi_B(a^{[2]}b)a^{[1]},$$

by Proposition 3.4.1.(i). Applying *, we see that

$$\varphi_A(b^*_{(1)}S_A(S_A(a)^*))b^*_{(0)} = \varphi_B(b^*a^{[2]*})a^{[1]*}.$$

Since the left hand side equals $\varphi_B(b^*(S_A(a)^*)^{[1]})(S_A(a)^*)^{[2]}$ by Proposition 3.4.1.(ii), we get that $(a^{[1]})^* \otimes (a^{[2]})^* = (S_A(a)^*)^{[2]} \otimes (S_A(a)^*)^{[1]}$ by the faithfulness of φ_B. This then becomes $S_B^{-1}((a_{[1]})^*) \otimes (a_{[2]})^* = (S_A(a)^*)_{[2]} \otimes S_C((S_A(a)^*)_{[1]})$. Applying S_B to the first leg and using the identity *xiii*) in Proposition 3.6.4, we arrive at the identity stated in the proposition.

□

Let B be a right *-Galois object for a *-algebraic quantum group. We show now that also the invariant functional ψ_B is positive, possibly after multiplying with a scalar. As for the *-algebraic quantum groups themselves, this is a non-trivial statement. We again do this by using a diagonalizability argument.

For instance, take $b \in B$ and choose $b' \in B$ with $\varphi_B(b') = 1$. Write $b \otimes b'$ as a sum

$$b \otimes b' = \sum_i b_i a_i^{[1]} \otimes a_i^{[2]}$$

for certain $b_i \in B$ and $a_i \in A$. Write $a_i = \sum_j a_{ij}$ with the a_{ij} eigenvectors for left multiplication with δ_A. Then by Proposition 3.2.4 and Proposition

3.5.4.*iv*), we have

$$\begin{array}{rcl} b\delta_B^n & = & \sum_i \varphi_B(a_i^{[2]})b_i a_i^{[1]}\delta_B^n \\ & = & \sum_i \varphi_B(\delta_B^n(\delta_A^{-n}a_i)^{[2]})b_i(\delta_A^{-n}a_i)^{[1]} \\ & \in & \mathrm{Span}\{\omega(a_{ij}^{[2]})b_i a_{ij}^{[1]} \mid \omega \in B^*\}, \end{array}$$

showing that $\mathrm{Span}\{b\delta_B^n \mid n \in \mathbb{Z}\}$ is finite-dimensional. The same technique shows that $\mathrm{Span}\{\delta_B^n b \mid n \in \mathbb{Z}\}$ is finite-dimensional.

Now B becomes a pre-Hilbert space by the inner product $\langle b, b'\rangle_B := \varphi_B(b'^*b)$, using the positivity, self-adjointness and faithfulness of φ_B as we did for *-algebraic quantum groups. Since $\overline{\psi_B} : b \to \overline{\psi_B(b^*)}$ is also an invariant functional on B, as is easily checked, we can replace ψ_B by $\psi_B + \overline{\psi_B}$ of $i(\psi_B - \overline{\psi_B})$ to obtain a *hermitian* invariant functional, which we will then take as our new ψ_B. Since $\sigma_B(\delta_B) = \delta_B$, the hermitianess of ψ_B implies that $\delta_B^* = \delta_B$, and moreover, that left and right multiplication with δ_B are self-adjoint with respect to the scalar product $\langle\,\cdot\,,\,\cdot\,\rangle_B$ on B. Since left and right multiplication commute, we obtain:

Theorem 3.9.4. *There exists a basis $\{b_i\}$ of B with the b_i joint eigenvectors for left and right multiplication with δ_B.*

Now for b, b' eigenvectors for left multiplication with δ_B with respective eigenvalues λ and λ', we can use that

$$(\varphi_B \otimes \psi_A)((b^* \otimes 1)\alpha_B(b'^*b')(b \otimes 1)) = \lambda'\lambda^{-1}\varphi_B(b^*b)\varphi_B(b'^*b')$$

to conclude that $\lambda^{-1}\lambda'$ is positive. After possibly multiplying ψ_B with -1 (and hence changing δ_B to $-\delta_B$), this implies that δ_B is positive. In particular, this shows again that δ_B is of the form $(\delta_B^{1/2})^2$ for some self-adjoint invertible element $\delta_B^{1/2} \in M(B)$. If we then choose $b' \in B$ with $\varphi_B((\delta_B^{-1/2}b')^*\delta_B^{-1/2}b') = 1$, we have for any $b \in B$ that

$$\begin{array}{rcl} \psi_B(b^*b) & = & \psi_B(b^*b)\varphi_B(b'^*\delta_B^{-1}b') \\ & = & \varphi_B(b'^*b_{(0)}^*b_{(0)'}b')\psi_A(b_{(1)}^*b_{(1)'}) \\ & \geqslant & 0, \end{array}$$

showing

Corollary 3.9.5. *There exists a positive invariant functional ψ_B on B.*

Note that the diagonizability of δ_B was really only used to find *one* element $b \in B$ with $\varphi_B(b^*\delta_B b) \neq 0$.

Chapter 4

Linking algebraic quantum groupoids

The theory of the previous chapter was concerned with Galois objects, which were algebras with a special coaction on them by an algebraic quantum group. In this chapter, we lift to the situation of algebraic quantum groups Proposition 1.3.10, i.e. we show that from any such a Galois object we can construct a new algebraic quantum group, and even more, that we have a natural coaction of this new quantum group on the original algebra, making it into a bi-Galois object. Our method of proof however is distinct from the Hopf-algebraic proof, since it is completely based on duality reasonings (which are not available for general Hopf algebras). For completeness, we also abstractly characterize the objects which can be considered to be the duals of bi-Galois objects, namely the linking algebraic quantum groupoids. After some brief discussion concerning the situation for *-algebraic quantum groups, we end with an example, which, although it takes place in the setting of Hopf algebras, and thus fits in the framework of Galois theory for Hopf algebras, at least produces new[1] examples of infinite-dimensional Hopf algebras with integrals.

Remark: In the paper [19], we added some categorical results concerning the categorical equivalence associated to a bi-Galois object, but we will not include this discussion here. There are several reasons for this. One of them is that the results of [19] are only partial: we constructed from a bi-Galois object a monoidal equivalence of unital module categories[2], but we

[1]as far as we know

[2]This is certainly easy for algebraic quantum groups, but we payed more attention to

did not consider the question of how to reconstruct a bi-Galois object from a monoidal equivalence (if this is possible at all without further, maybe unnatural conditions). Another reason is that we do not think that, at this stage, the categorical viewpoint would add any extra value to the discussion. We have included it in the first chapter by way of motivation, since in that case, we can then say *precisely* what the (big) invariant is which is preserved under the (co-)monoidal (co-)Morita equivalence, namely some monoidal category. Because of our lack of a reconstruction theorem, this 'invariant' becomes less clear to characterize in the case of algebraic quantum groups, and certainly in the case of locally compact quantum groups, to be considered in the second part of our thesis.

4.1 Linking algebraic quantum groupoids and bi-Galois objects

Definition 4.1.1. *We call* linking multiplier weak Hopf (*-)algebra *a triple* (E, e, Δ_E) *consisting of a non-degenerate linking (*-)algebra* (E, e)*, together with a coassociative u.e. (*-)homomorphism* $\Delta_E : E \to M(E \odot E)$ *for which* $\Delta_E(e) = e \otimes e$ *and* $\Delta_E(1_E - e) = (1_E - e) \otimes (1_E - e)$*, and such that* $A = eEe$ *and* $D = (1_E - e)E(1_E - e)$*, together with the restrictions of* Δ_E*, become multiplier Hopf (*-)algebras.*

We make some comments about this definition. First remark that the coassociativity statement about Δ_E makes sense by the fact that the tensor product of two u.e. maps is again u.e. Also, it is easily seen that Δ_E is in fact u.e. with respect to $(e \otimes e) + ((1_E - e) \otimes (1_E - e))$. Next, because $E \odot E$ is a non-degenerate linking algebra between $A \odot A$ and $D \odot D$, we know that $(e \otimes e)M(E \odot E)(e \otimes e)$ can be identified with $M(A \odot A)$ by Lemma 2.2.6, so there is no ambiguity concerning the statement 'restricting Δ_E to A'. Finally, this definition is not very natural, since we do not define a linking multiplier weak Hopf algebra as a multiplier weak Hopf algebra satisfying certain properties. The reason for this is simple: there is as of yet no such notion, although there is some work in progress on it. Instead of developing it, we have rather opted for an ad hoc approach.

the *-algebraic context, since we can then consider equivalence of the more specialized monoidal *-categories of *-representations in pre-Hilbert spaces. Since one wants all natural transformations adapted to this *-structure, one has to do some more work.

On the other hand, there are some obvious properties which one knows should hold for any multiplier weak Hopf algebra. We collect them in the following proposition. We continue to use Sweedler notation for the comultiplication.

Proposition 4.1.2. *Let (E, e) be a linking multiplier weak Hopf algebra.*

- *The comultiplication Δ_E has range in $M_{1;2}(E \odot E)$.*
- *The map $E \odot E \to E \odot E : x \otimes y \to \Delta_E(x)(1 \otimes y)$ restricts to bijections $E_{ij} \otimes E_{jk} \to E_{ij} \otimes E_{ik}$ (and similarly for all other maps of this form).*
- *There exists a unique functional*

$$\varepsilon_E : E \to k,$$

called the co-unit, *such that*

$$(\varepsilon_E \otimes \iota_E)\Delta_E(x_{ij}) = x_{ij} = (\iota_E \otimes \varepsilon_E)\Delta_E(x_{ij}) \qquad \text{for } x_{ij} \in E_{ij}.$$

Moreover, this counit satisfies

$$\varepsilon_E(x_{ij} \cdot x'_{jk}) = \varepsilon_E(x_{ij})\varepsilon_E(x'_{jk}) \qquad x_{ij} \in E_{ij}, x'_{jk} \in E_{jk}.$$

- *There exists a unique map*

$$S_E : E \to E,$$

called the antipode, *such that*

$$M_E((S_E \otimes \iota_E)(\Delta_E(x_{ij})(1_E \otimes x'_{jk}))) = \varepsilon_E(x_{ij})x'_{jk}$$

for all $x_{ij} \in E_{ij}, x'_{jk} \in E_{jk}$, and

$$M_E((\iota_E \otimes S_E)((x_{ij} \otimes 1_E)\Delta_E(x'_{jk}))) = \varepsilon_E(x'_{jk})x_{ij}$$

for all $x_{ij} \in E_{ij}, x'_{jk} \in E_{jk}$. Moreover, this map will then be an anti-automorphism, and $S_E(E_{ij}) = E_{ji}$.

Proof. Remark that Δ_E restricts to linear maps $E_{ij} \to M(E_{ij} \odot E_{ij})$, which we will denote as Δ_{ij}.

We have then that

$$\begin{aligned}
\Delta_{ij}(E_{ij})(1\otimes E_{jk}) &= \Delta_{ij}(E_{ij}\cdot E_{jj})(1\otimes (E_{jj}\cdot E_{jk}))\\
&= \Delta_{ij}(E_{ij})\cdot(\Delta_{jj}(E_{jj})(1\otimes E_{jj}))\cdot(1\otimes E_{jk})\\
&= \Delta_{ij}(E_{ij})\cdot(E_{jj}\odot(E_{jj}\cdot E_{jk}))\\
&= \Delta_{ij}(E_{ij})\cdot(E_{jj}\odot E_{jk}).
\end{aligned}$$

Now

$$\Delta_{i1}(E_{i1})(E_{1j}\odot E_{1k}) = \Delta_{i2}(E_{i2})(E_{2j}\odot E_{2k}):$$

for example $\subseteq$ holds since $E_{i1} = E_{i2}\cdot E_{21}$, hence

$$\begin{aligned}
\Delta_{i1}(E_{i1})(E_{1j}\odot E_{1k}) &= \Delta_{i2}(E_{i2})\Delta_{21}(E_{21})(E_{1j}\odot E_{1k})\\
&\subseteq \Delta_{i2}(E_{i2})(E_{2j}\odot E_{2k}).
\end{aligned}$$

The u.e. property of Δ_E, together with this last fact, then implies that

$$\Delta_{ij}(E_{ij})\cdot(E_{jj}\odot E_{jk}) = E_{ij}\odot E_{ik}.$$

Hence $\Delta_{ij}(E_{ij})(1\otimes E_{jk}) = E_{ij}\odot E_{ik}$, and the maps stated in the second item are all surjective.

Now suppose that $x_{ij,p}\in E_{ij}$ and $x'_{jk,p}\in E_{jk}$ are such that

$$\sum_p \Delta_{ij}(x_{ij,p})(1\otimes x'_{jk,p}) = 0.$$

Taking an arbitrary $z_{ji}\in E_{ji}$ and $w_{kj}\in E_{kj}$, we see that also

$$\sum_p \Delta_{jj}(z_{ji}x_{ij,p})(1\otimes x'_{jk,p}w_{kj}) = 0,$$

and hence

$$\sum_p z_{ji}x_{ij,p}\otimes x'_{jk,p}w_{kj} = 0,$$

by definition of a multiplier Hopf algebra. Multiplying the first leg to the left with an arbitrary element of E_{lj}, and using that $E_{lj}\cdot E_{ji} = E_{li}$, we see that

$$\sum_p (z\cdot x_{ij,p})\otimes x'_{jk,p}w_{kj} = 0$$

for an arbitrary $z\in E$, hence

$$\sum_p x_{ij,p}\otimes x'_{jk,p}w_{kj} = 0,$$

by non-degeneracy of $E \odot E$. A similar argument applied to the second leg lets us conclude that

$$\sum_p x_{ij,p} \otimes x'_{jk,p} = 0.$$

Hence all maps stated in the second item are bijective. Then by symmetry, the first two items are proven.

We now construct the counit as in the third item. In fact, most of the work has already been done in the first chapter. Indeed: the beginning of the proof of Proposition 1.2.18 can be copied ad verbum, and lets us conclude that there exists ε_B on B such that $(\varepsilon_B \otimes \iota_B)\Delta_B = (\iota_B \otimes \varepsilon_B)\Delta_B = \iota_B$. A similar map ε_C then exists on C by symmetry, and we define ε_E as the direct sum of the functionals $\varepsilon_D, \varepsilon_C, \varepsilon_B$ and ε_A, where the first and last map are the counits of resp. D and A.

Also the 'bimodularity' of ε_E is then partially contained in the proof of Proposition 1.2.18. The only thing which does not follow immediately is if $\varepsilon_D(bc) = \varepsilon_B(b)\varepsilon_C(c)$ (and the symmetric counterpart with respect to A), but this proof is in fact completely similar:

$$\begin{aligned} db_{(1)}c_{(1)}d'\varepsilon_D(b_{(2)}c_{(2)}) &= d(bc)_{(1)}d'\varepsilon_D((bc)_{(2)}) \\ &= dbcd' \\ &= db_{(1)}cd'\varepsilon_B(b_{(2)}) \\ &= db_{(1)}c_{(1)}d'\varepsilon_B(b_{(2)})\varepsilon_C(c_{(2)}), \end{aligned}$$

which implies $\varepsilon_D(bc) = \varepsilon_B(b)\varepsilon_C(c)$ for all $b \in B$ and $c \in C$ by bijectivity of the maps in the second item.

We move on to the antipode. Denote $\tilde{C} := \mathrm{Hom}_D({}_DB, {}_DD)$. We can identify C with a subspace of $\tilde{C}$, letting C act on B by right multiplication. We will also write the action of $\tilde{C}$ on B as right multiplication: if $x \in \tilde{C}$ and $b \in B$, we write

$$bx = b \cdot x := x(b).$$

Then the proof of Proposition 1.2.18 can still be copied up to some point, to conclude that we have a map $S_B : B \to \tilde{C}$, such that we have $db_{(1)}S_B(b_{(2)}) = \varepsilon_B(b)d$ and $(b'S_B(b_{(1)})) \cdot b_{(2)} \cdot a = \varepsilon_B(b)b'a$. We want to show that S_B has range in C.

First remark that $\mathrm{Hom}_D({}_DB, {}_DD)$ is a left A-module, by defining $b(a\cdot x) := (ba)\cdot x$ for $x \in \tilde{C}$. This extends the natural right A-module structure on C. Then for $c \in C$, and $b, b', b'' \in B$, we have

$$\begin{aligned} b''((cb')\cdot S_B(b)) &= (b''cb')S_B(b) \\ &= (b''c)\cdot(b'S_B(b)) \\ &= b''\cdot(c\cdot(b'S_B(b))). \end{aligned}$$

Hence $(cb')\cdot S_B(b) = c\cdot(b'S_B(b))$. Since $C\cdot B = A$, we see that $AS_B(B) \subseteq C$. We want to show now that $S_A(a)\cdot S_B(b) = S_B(ba)$. It is easy to see that $ba_{(1)}S_A(a_2) = \varepsilon_A(a)b$, since $B = B\cdot A$. Hence, for $d \in D, b \in B$ and $a \in A$, we compute

$$\begin{aligned} db_{(1)}a_{(1)}S_B(b_{(2)}a_{(2)}) &= d(ba)_{(1)}S_B((ba)_{(2)}) \\ &= \varepsilon_B(ba)d \\ &= \varepsilon_B(b)\varepsilon_A(a)d \\ &= \varepsilon_A(a)db_{(1)}S_B(b_{(2)}) \\ &= db_{(1)}a_{(1)}S_A(a_{(2)})S_B(b_{(2)}). \end{aligned}$$

From this, it follows that $bS_B(b'a) = bS_A(a)S_B(b)$ for all $b, b' \in B$ and $a \in A$, using bijectivity of the maps of the second item. Hence $S_A(a)S_B(b) = S_B(ba)$. So we have in fact $S_B(B) = S_B(B\cdot A) = A\cdot S_B(B) \subseteq C$. One can then also easily prove that $S_B(d\cdot b) = S_B(b)S_D(d)$, and that $S_B(b_{(1)})b_{(2)}a = \varepsilon_B(b)a$.

By similar reasonings, one constructs an antipode $S_C : C \to B$, and it is then not hard to show that the direct sum of S_D, S_C, S_B and S_D, which is a map $E \to E$, is an anti-homomorphism, satisfying the antipode conditions as in the fourth item.

Finally, we show that S_E is bijective. It is sufficient to show that S_B is bijective. Suppose first that $S_B(b) = 0$. Multiplying to the left with $S_C(c)$ for some $c \in C$, we get that $S_D(bc) = 0$, hence $bc = 0$. Since c was arbitrary, the non-degeneracy of E easily implies that $b = 0$. So S_E is injective. Now take an arbitrary $c \in C$, and choose $c_i, c_i' \in C, b_i \in B$ such that $c = \sum_i c_ib_ic_i'$. Write $S_A^{-1}(c_ib_i) = \sum_j c_{ij}b_{ij}$. Then $c = \sum_{i,j} S_A(c_{ij}b_{ij})c_i'$, which equals $\sum_{i,j} S_B(b_{ij})S_C(c_{ij})c_i'$. Since $S_B(B)D = S_B(B)$, we find that $c \in S_B(B)$. Hence S_B is bijective.

The uniqueness statements concerning the counit and antipode map are easy to establish, and the proof will be omitted.

□

Definition 4.1.3. *If A and D are two multiplier Hopf (*-)algebras, we call* linking multiplier weak Hopf (*-)algebra between A and D *a linking multiplier weak Hopf (*-)algebra (E,e), together with (*-)isomorphisms $E_{22} \underset{\Phi_A}{\cong} A$ and $E_{11} \underset{\Phi_D}{\cong} D$ as multiplier Hopf (*-)algebras. When A and D are actually (*-)algebraic quantum groups, we call (E,e) a linking* (*-)algebraic quantum groupoid *between A and D.*

When A and D are two multiplier Hopf (-)algebras, we call them* comonoidally (*-)Morita equivalent *if there exists a linking multiplier weak Hopf (*-)algebra between them.*

We will follow conventions as for linking Hopf algebras between, and not explicitly write the Φ_A and Φ_D.

By definition, the algebras underlying two comonoidally Morita equivalent multiplier Hopf algebras are non-degenerately Morita equivalent. Similarly as for Hopf algebras, we then have the notion of an identity linking multiplier weak Hopf algebra, the inverse of a linking multiplier weak Hopf algebra, and the composition of two linking multiplier weak Hopf algebras. The first two constructions are trivial. As for the construction of the composition, let E_1 and E_2 be linking multiplier weak Hopf algebras between resp. E_{22} and E_{11}, and E_{33} and E_{22}. Consider the associated 3×3-linking algebra $E = (E_{ij})_{i,j\in\{1,2,3\}}$. Then E_1, E_2 and their composite linking algebra E_3 can all be imbedded by an u.e. map into $\tilde{E}$. Then for example $M(E_{1,12})$ will get sent to $M(E_{12})$. The same holds true for tensor products. Hence if $x_{12} \in E_{12}$ and $y_{23} \in E_{23}$, we can compose $\Delta_{12}(x_{12})$ and $\Delta_{23}(y_{23})$ inside $M(\tilde{E})$, and obtain an element of $M(E_{13} \odot E_{13}) = M(E_{3,12} \odot E_{3,12})$. Since E_{13} also equals $E_{12} \underset{E_{22}}{\odot} E_{23}$, and since all algebras have local units, it is not difficult to see that

$$\Delta_{13} : E_{13} \to M(E_{13} \odot E_{13}) : x_{12} \cdot y_{23} \to \Delta_{12}(x_{12})\Delta_{23}(y_{23})$$

extends to a well-defined map $E_{3,12} \to M(E_{3,12} \odot E_{3,12})$. Similarly, one constructs a map $\Delta_{31} : E_{3,21} \to M(E_{3,21} \odot E_{3,21})$, and we can then combine these with the comultiplications of E_{11} and E_{33} to obtain a map $\Delta_{E_3} : E_3 \to M(E_3 \odot E_3)$. We leave it to the reader to check that Δ_{E_3}

is a coassociative u.e. homomorphism, making E_3 into a weak linking multiplier Hopf algebra between E_{11} and E_{33}.

In general, a (non-degenerate) linking algebra is determined by its C, B and A-part, but not by its B and A-part. The situation is different for linking multiplier weak Hopf algebras.

Proposition 4.1.4. *Let E_1 and E_2 be two linking multiplier weak Hopf algebras, and suppose there are linear isomorphisms $\Phi_{22} : A_1 \to A_2$ and $\Phi_{12} : B_1 \to B_2$, such that $\Phi_{12}(ba) = \Phi_{12}(b)\Phi_{22}(a)$ and*

$$(\Phi_{12} \otimes \Phi_{12})(\Delta_{B_1}(b)(1 \otimes a)) = \Delta_{B_2}(\Phi_{12}(b))(1 \otimes \Phi_{22}(a))$$

for all $b \in B_1, a \in A_1$. Then E_1 and E_2 are isomorphic linking multiplier weak Hopf algebras, by an isomorphism Φ extending the Φ_{12} and Φ_{22}.

Proof. Define $\Phi_{21} := S_{B_2} \circ \Phi_{12} \circ S_{C_1}$ and

$$\Phi_{22} : D_1 \to D_2 : b \cdot c \to \Phi_{12}(b)\Phi_{21}(c).$$

Then an easy argument shows that the direct sum of the Φ_{ij} provides the wanted isomorphism Φ.

□

One can also define the notion of a comonoidal right Morita module for a multiplier Hopf algebra. This theory if developed in [22]. We will not be concerned with this here, but we wish to remark that, unlike the theory of comonoidal right Morita modules for *Hopf* algebras, there is an extra condition to be imposed on right comonoidal Morita modules to be able to perform the reflection technique of Proposition 1.2.18, which then pushes the definition already further into the direction of a linking weak multiplier Hopf algeba. It is shown further in [23] that there is a *concrete* duality between right comonoidal Morita modules for some algebraic quantum group, and Galois objects for its dual. We will not prove this correspondence here, but parts of it will appear in the ensuing discussion.

Dual to the notion of a linking weak multiplier Hopf algebra, we should introduce the notion of a co-linking weak multiplier Hopf algebra. However, we will restrict ourselves to defining the basic constituent of this last structure.

Definition 4.1.5. *Let A and D be two multiplier Hopf (*-)algebras. A* bi-(*)Galois object *between A and D (or D-A-bi-(*-)Galois object) consists of a triple (B, γ_B, α_B) such that (B, γ_B) is a left D-(*-)Galois object, (B, α_B) is a right A-(*-)Galois object, and γ_B and α_B commute:*

$$(\gamma_B \otimes \iota_A)\alpha_B = (\iota \otimes \alpha_B)\gamma_B.$$

Again, it is not clear whether Proposition 1.3.10 continues to hold in the general setting of multiplier Hopf algebras. As mentioned in the beginning of this chapter, we will however show in the following two sections that one *can* construct bi-Galois objects from Galois objects for algebraic quantum groups.

4.2 From Galois objects to linking algebraic quantum groupoids

Given a right Galois object for an algebraic quantum group, we want to construct from it a linking algebraic quantum groupoid (and, in particular, a new algebraic quantum group, given as the upper left corner of the linking algebraic quantum groupoid). *For the rest of this section, A will always denote an algebraic quantum group, and (B, α_B) a right A-Galois object.* We will also continue to use the notation introduced in the previous chapter without further comment.

Definition 4.2.1. *Let B be a right A-Galois object. The* restricted dual *of B is the vector space $\widehat{B} = \{\varphi_B(\,\cdot\, b) \mid b \in B\}$ inside the dual B^* of B.*

We have shown

$$\begin{aligned}\widehat{B} &= \{\varphi_B(b\,\cdot\,) \mid b \in B\}\\ &= \{\psi_B(\,\cdot\, b) \mid b \in B\}\\ &= \{\psi_B(b\,\cdot\,) \mid b \in B\}\end{aligned}$$

in Theorems 3.3.2 and 3.4.3, so as for algebraic quantum groups, all natural definitions for a restricted dual give us the same space. We will denote the elements of $\widehat{B}$ (or B^*) as $\omega_{12}, \omega'_{12}, \ldots$, or, if we consider an indexed family, as ω^i_{12}. The reason for this is that $\widehat{B}$ will later be treated as the upper right corner of a linking algebra.

Let $\widehat{A}$ be the dual of A. For the same reason, we will now denote elements of $\widehat{A}$ as $\omega_{22}, \omega'_{22}, \ldots$. As already mentioned in the previous chapter, we have a left unital $\widehat{A}$-module structure on B, induced by α_B, by putting

$$\omega_{22} \cdot b = (\iota_B \otimes \omega_{22})(\alpha_B(b))$$

for $b \in B$ and $\omega_{22} \in \widehat{A}$. This leads to a *right* $\widehat{A}$-module structure on the dual B^*, by putting

$$(\omega_{12} \cdot \omega_{22})(b) = \omega_{12}(\omega_{22} \cdot b)$$

for $\omega_{12} \in B^*, \omega_{22} \in \widehat{A}$ and $b \in B$.

Lemma 4.2.2. *The right $\widehat{A}$-module structure on B^* restricts to a unital right $\widehat{A}$-module structure on $\widehat{B}$.*

Proof. Take $b, b' \in B$ and $\omega_{22} \in \widehat{A}$. Then

$$\begin{aligned}(\psi_B(\,\cdot\, b) \cdot \omega_{22})(b') &= \omega_{22}(\psi_B(b'_{(0)}b)b'_{(1)}) \\ &= (\omega_{22} \circ S_A)(\psi_B(b'b_{(0)})b_{(1)}) \\ &= (\psi_B(\,\cdot\,(S_{\widehat{A}}(\omega_{22}) \cdot b)))(b'),\end{aligned}$$

by using Proposition 3.3.1. By the surjectivity of $T_{\alpha_B,2}$ (see the discussion after Definition 2.5.1, this will be a *unital* right $\widehat{A}$-module. □

The space $\widehat{B}$ also carries a *natural* A^*-valued k-bilinear form, determined by

$$[\,\omega_{12}, \omega'_{12}\,]_{\widehat{A}}(a) = (\omega_{12} \otimes \omega'_{12})(\tilde{\beta}_A(a)), \qquad \omega_{12}, \omega'_{12} \in \widehat{B}, a \in A,$$

where β_A was defined in Proposition 3.1.2.

Proposition 4.2.3. *Let B be a right A-Galois object, and $[\,\cdot\,,\,\cdot\,]_{\widehat{A}}$ as above. Then $[\,\cdot\,,\,\cdot\,]_{\widehat{A}}$ is non-degenerate, has $\widehat{A}$ as its range, and is right $\widehat{A}$-linear, i.e. for all $\omega_{12}, \omega'_{12} \in \widehat{B}$ and $\omega_{22} \in \widehat{A}$,*

$$[\,\omega_{12}, \omega'_{12} \cdot \omega_{22}\,]_{\widehat{A}} = [\,\omega_{12}, \omega'_{12}\,]_{\widehat{A}} \cdot \omega_{22}.$$

Proof. If $\omega_{12} = \varphi_B(b\,\cdot\,)$, then

$$\begin{aligned}[\,\omega_{12}, \omega'_{12}\,]_{\widehat{A}}(a) &= \varphi_B(ba^{[1]})\omega'_{12}(a^{[2]}) \\ &= \varphi_A(b_{(1)}S_A(a))\omega'_{12}(b_{(0)}) \\ &= (\psi_A(\,\cdot\, S_A^{-1}(\omega'_{12}(b_{(0)})b_{(1)})))(a),\end{aligned}$$

using the second formula of Proposition 3.4.1. So the form takes values in $\widehat{A}$. The surjectivity of the Galois map gives that the range of the form is the whole of $\widehat{A}$. The faithfulness of the invariant functional on B shows that the bracket is non-degenerate. Finally,

$$\begin{aligned}([\,\omega_{12},\omega'_{12}\,]_{\widehat{A}}\cdot\omega_{22})(a) &= (\omega_{12}\otimes\omega'_{12}\otimes\omega_{22})((\tilde{\beta}_A\otimes\iota_A)\Delta_A(a))\\ &= (\omega_{12}\otimes\omega'_{12}\otimes\omega_{22})((\iota_A\otimes\alpha_A)\tilde{\beta}_A(a))\\ &= [\,\omega_{12},\omega'_{12}\cdot\omega_{22}\,]_{\widehat{A}}(a),\end{aligned}$$

where we have used Lemma 3.7.2. So the bracket is right $\widehat{A}$-linear.

□

We use this bracket to construct a non-degenerate linking algebra which has $\widehat{A}$ as its lower right corner. First, we identify $\widehat{A}\subseteq \mathrm{End}_{\widehat{A}}(\widehat{A}_{\widehat{A}})$ as left multiplication operators, and also $\widehat{B}\subseteq \mathrm{Hom}_{\widehat{A}}(\widehat{A}_{\widehat{A}},\widehat{B}_{\widehat{A}})$ as 'left multiplication operators' (which will be faithful, for example by using the unitality of B as a left $\widehat{A}$-module *and* then the fact that $\widehat{A}$ has right local units). Then define

$$\widehat{C}:=\{[\omega_{12},\,\cdot\,]_{\widehat{A}}\mid \omega_{12}\in\widehat{B}\}\subseteq \mathrm{Hom}_{\widehat{A}}(\widehat{B}_{\widehat{A}},\widehat{A}_{\widehat{A}}),$$

where the inclusion at the end follows from Proposition 4.2.3, and put

$$\widehat{D}:=\widehat{B}\cdot\widehat{C}\subseteq \mathrm{End}_{\widehat{A}}(\widehat{B}_{\widehat{A}}),$$

where the dot denotes composition. We group them together into the algebra

$$\widehat{E}:=\begin{pmatrix}\widehat{D} & \widehat{B}\\ \widehat{C} & \widehat{A}\end{pmatrix}\subseteq \mathrm{End}_{\widehat{A}}(\begin{pmatrix}\widehat{B}\\ \widehat{A}\end{pmatrix}_{\widehat{A}}).$$

We will write elements of $\widehat{C}$ as ω_{21}, and elements of $\widehat{D}$ as ω_{11}.

Lemma 4.2.4. *The map*

$$S_{\widehat{B}}:\widehat{B}\to\widehat{C}:\omega_{12}\to[\omega_{12},\,\cdot\,]_{\widehat{A}}$$

is a bijection.

Proof. Suppose that $[\omega_{12},\,\cdot\,]_{\widehat{A}}=0$. Then $(\omega_{12}\otimes\omega'_{12})(\tilde{\beta}_A(a))=0$ for all $a\in A$ and $\omega'_{12}\in\widehat{B}$. By the surjectivity of the map

$$A\odot B\to B\odot B: a\otimes b\to a^{[1]}\otimes a^{[2]}b,$$

this means $\omega_{12}(b)=0$ for all $b\in B$, hence $\omega_{12}=0$.

□

We can use the previous lemma to view $\widehat{C}$ as functionals on C (which, we recall, is nothing else but B^{op}).

Proposition 4.2.5. *Let B be a right A-Galois object, and $\widehat{C}$ as above. There is a natural non-degenerate pairing*

$$\widehat{C} \times C \to k : ([\omega_{12}, \cdot\,]_{\widehat{A}}, b^{op}) \to \omega_{12}(b).$$

Proof. By the previous lemma, the map is well-defined. The non-degeneracy follows immediately from the faithfulness of φ_B. □

We write the pairing between $\omega_{21} \in \widehat{C}$ and $c \in C$ as $\omega_{21}(c)$ of course. In fact, it is easy to see then that $\widehat{C}$ is just the space of functionals on C of the form $\varphi_C(\,\cdot\, c)$, with $c \in C$, hence coincides with the restricted dual of the left Galois object C. So there is no conflict of notation.

Lemma 4.2.6. *By composition of linear maps, the space $\widehat{C}$ becomes a unital left $\widehat{A}$-module, and then*

$$(\omega_{22} \cdot \omega_{21})(c) = (\omega_{22} \otimes \omega_{21})(\gamma_C(c)).$$

Proof. Take $\omega_{21} \in \widehat{C}$, $\omega_{12} \in \widehat{B}$, $\omega_{22} \in \widehat{A}$ and $a \in A$. Then by definition of the external comultiplication β_A (see Definition 3.5.3), we get

$$\begin{array}{rcl}((\omega_{22} \cdot \omega_{21})(\omega_{12}))(a) & = & (\omega_{22} \cdot [S_{\widehat{B}}^{-1}(\omega_{21}), \omega_{12}]_{\widehat{A}})(a) \\ & = & (\omega_{22} \otimes \omega_{21} \otimes \omega_{12})((\iota_A \otimes \beta_A)\Delta_A(a)) \\ & = & (\omega_{22} \otimes \omega_{21} \otimes \omega_{12})((\gamma_C \otimes \iota_B)\beta_A(a)) \\ & = & (((\omega_{22} \otimes \omega_{21}) \circ \gamma_C)(\omega_{12}))(a),\end{array}$$

which proves the formula

$$(\omega_{22} \cdot \omega_{21})(c) = (\omega_{22} \otimes \omega_{21})(\gamma_C(c)).$$

Then the fact that $\widehat{C}$ is a unital left $\widehat{A}$-module follows (for example) by symmetry from Lemma 4.2.2.

□

By these results, it is clear that the space $\widehat{C}$ can be identified with the restricted dual space for the *left* Galois object (C, γ_C) *as a left $\widehat{A}$-module*, and that the space $\widehat{B}$ *as constructed from the left Galois object* (C, γ_C), with all the extra structure, can also be identified with the one considered up to now. So we can treat $\widehat{B}$ and $\widehat{C}$, as resp. right and left $\widehat{A}$-module, on an equal, symmetric footing.

Proposition 4.2.7. *The couple* $(\widehat{E}, \begin{pmatrix} 0 & 0 \\ 0 & 1_{\widehat{A}} \end{pmatrix})$ *is a non-degenerate linking algebra between* $\widehat{A}$ *and* $\widehat{D}$.

Proof. We first show that $\widehat{E}e\widehat{E} = \widehat{E}$ and $\widehat{E}(1-e)\widehat{E} = \widehat{E}$. This follows from the fact that $\widehat{A}$ is idempotent (so $\widehat{A}\cdot\widehat{A} = \widehat{A}$), that $\widehat{B}$ is a *unital* right $\widehat{A}$-module (which gives that $\widehat{B}\cdot\widehat{A} = \widehat{B}$), from Proposition 4.2.3 and the remark just before it (which shows that $\widehat{C}\cdot\widehat{B} = \widehat{A}$), from Lemma 4.2.6 (which shows that $\widehat{A}\cdot\widehat{C} = \widehat{C}$), and from the definition of $\widehat{D}$ (which gives $\widehat{B}\cdot\widehat{C} = \widehat{D}$). From these pieces, *all* the equalities $\widehat{E}_{ij}\cdot\widehat{E}_{jk} = \widehat{E}_{ik}$ are easily derived.

We still have to show that $\widehat{E}$ is non-degenerate. But since $\widehat{E}$ is faithfully represented as linear maps on $\begin{pmatrix} \widehat{B} \\ \widehat{A} \end{pmatrix}$, it is easy to see that if $x \in \widehat{E}$ and $xy = 0$ for all $y \in \widehat{E}$, then $x = 0$. Similarly, $\widehat{E}$ is faithfully represented as linear maps on $(\widehat{C} \;\; \widehat{A})$ by *right* multiplication (since we have shown that the roles of B and C are symmetrical, or by a simple direct verification), and then $x \in \widehat{E}$ and $yx = 0$ for all $y \in \widehat{E}$ leads to $x = 0$.

□

In particular, $\widehat{D}$ is a non-degenerate algebra. However, using only that $\widehat{A}$ is non-degenerately Morita equivalent to $\widehat{D}$, we can not say more, as there is in general no reason to expect that the property of 'having local units' is preserved under non-degenerate Morita equivalence. So the following lemma shows in how well-behaving a situation we are.

Lemma 4.2.8. *For any finite collection* $\omega_{12}^i \in \widehat{B}$ *there exists* $\omega_{11} \in \widehat{D}$ *with* $\omega_{11}\cdot\omega_{12}^i = \omega_{12}^i$. *Likewise, for any finite collection* $\omega_{21}^i \in \widehat{C}$ *there exists* $\omega_{11} \in \widehat{D}$ *with* $\omega_{21}^i\cdot\omega_{11} = \omega_{21}^i$.

Proof. Fix a finite collection of ω_{21}^i, and write $\omega_{12}^i = \varphi_B(\,\cdot\, b_i)$. We have to prove that there exist $\omega_{12}'^j$ and $\omega_{21}'^j \in \widehat{C}$ such that

$$\sum_j \omega_{12}'^j \cdot \omega_{21}'^j \cdot \omega_{12}^i = \omega_{12}^i.$$

This means that for any $b \in B$ and all i, we must have

$$\sum_j \omega'^{j}_{12}(b_{(0)})\omega'^{j}_{21}(b_{(1)[1]})\varphi_B(b_{(1)[2]}b_i) = \varphi_B(bb_i).$$

By using Proposition 3.4.1, our problem is thus equivalent to finding some $\omega'^{j}_{12}, \omega''^{j}_{12} \in \widehat{B}$ such that for all i and for all $b \in B$,

$$\sum_j \omega'^{j}_{12}(b_{(0)})\omega''^{j}_{12}(b_{i(0)})\varphi_A(b_{(1)}b_{i(1)}) = \varphi_B(bb_i).$$

Choose $b' \in B$ with $\varphi_B(b') = 1$. Put $\omega_{12} = \varphi_B(\,\cdot\, b')$ and choose $\omega'_{12} \in \widehat{A}$ such that $\omega'_{12}(b_{i(1)})b_{i(0)}b' \otimes b_{i(2)} = b_{i(0)}b' \otimes b_{i(1)}$ for all i (using that A is a unital left $\widehat{A}$-module, and that $\widehat{A}$ has local units). Put $\omega_{12(1)} \otimes (\omega_{12(2)} \cdot \omega'_{12}) = \sum_j \omega'^{j}_{12} \otimes \omega''^{j}_{12}$. Then we have

$$\begin{aligned}
\sum_j \omega'^{j}_{12}(b_{(0)})\omega''^{j}_{12}(b_{i(0)})\varphi_A(b_{(1)}b_{i(1)}) &= \omega_{12}(b_{(0)}b_{i(0)})\varphi_A(b_{(1)}b_{i(2)})\omega'_{12}(b_{i(1)}) \\
&= \omega_{12}(b_{(0)}b_{i(0)})\varphi_A(b_{(1)}b_{i(1)}) \\
&= \varphi_B(bb_i).
\end{aligned}$$

The second statement follows by symmetry.

□

Corollary 4.2.9. *The algebra $\widehat{D}$ has local units.*

By the discussion concerning composition and inverses of linking algebras in section 2.2, we have the following corollary (which is easily verified).

Corollary 4.2.10. *The natural projection*

$$\pi : \widehat{B} \underset{\widehat{A}}{\otimes} \widehat{C} \to \widehat{D} : \omega_{12} \underset{\widehat{A}}{\otimes} \omega_{21} \to \omega_{12} \cdot \omega_{21}$$

is bijective.

We can use this observation to construct an antipode $S_{\widehat{E}}$ and counit $\varepsilon_{\widehat{E}}$ on $\widehat{E}$ (although we will show only later that they satisfy the expected properties with respect to a still to be defined comultiplication). First, note that we have already defined a map $S_{\widehat{B}} : \widehat{B} \to \widehat{C}$. Denote further by $S^2_{\widehat{B}}$ the restriction of the transpose of S^2_B to $\widehat{B}$. By the 'invariance up to a scalar' of S_B^2 with respect to φ_B (Proposition 3.6.2 *xi*)), it is easy to see that $S_{\widehat{B}}^2$ is a map $\widehat{B} \to \widehat{B}$. By Lemma 4.2.4, we can also define

$$S_{\widehat{C}} : \widehat{C} \to \widehat{B} : [\omega_{12}, \cdot\,]_{\widehat{A}} \to S_{\widehat{B}}^2(\omega_{12}),$$

and then $S_{\widehat{B}}^2 = S_{\widehat{C}} \circ S_{\widehat{B}}$. In fact, we also have the expression

$$S_{\widehat{C}}(\omega_{21})(b) = \omega_{21}(S_B(b))$$

for $S_{\widehat{C}}$, which is easily verified.

Lemma 4.2.11. *For $\omega_{12} \in \widehat{B}, \omega_{21} \in \widehat{C}$ and $\omega_{22} \in \widehat{A}$, we have*

$$S_{\widehat{A}}(\omega_{21} \cdot \omega_{12}) = S_{\widehat{B}}(\omega_{12}) \cdot S_{\widehat{C}}(\omega_{21}),$$

$$S_{\widehat{B}}(\omega_{12} \cdot \omega_{22}) = S_{\widehat{A}}(\omega_{22}) \cdot S_{\widehat{B}}(\omega_{12}),$$

$$S_{\widehat{C}}(\omega_{22} \cdot \omega_{21}) = S_{\widehat{C}}(\omega_{21}) \cdot S_{\widehat{A}}(\omega_{22}).$$

Proof. Seeing $\widehat{A}$, $\widehat{B}$ and $\widehat{C}$ as functionals, using that the various compositions are duals (= transposes) of the maps β_A, α_B and γ_C, and that the antipodes $S_{\widehat{A}}, S_{\widehat{B}}$ and $S_{\widehat{C}}$ are dual to the antipodes on A, B and C, the identities follow from the fact that

$$\beta_A \circ S_A = (S_B \otimes S_C) \circ \beta_A^{\mathrm{op}},$$

$$\alpha_B \circ S_C = (S_C \otimes S_A) \circ \gamma_C^{\mathrm{op}}$$

and

$$\gamma_C \circ S_B = (S_A \otimes S_B) \circ \alpha_B^{\mathrm{op}},$$

where the first identity follows by Proposition 3.6.4.*xiii*), the second one follows from the definition of γ_C, and the third one follows from Proposition 3.6.2.*vii*).

□

Corollary 4.2.12. *There is a well-defined anti-automorphism*

$$S_{\widehat{D}} : \widehat{D} \to \widehat{D} : \omega_{12} \cdot \omega_{21} \to S_{\widehat{C}}(\omega_{21}) \cdot S_{\widehat{B}}(\omega_{12}).$$

Proof. This follows straightforwardly from Corollary 4.2.10 and the previous lemmas. □

We collect these antipodes together into a single anti-automorphism

$$S_{\widehat{E}} : \widehat{E} \to \widehat{E} : \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix} \to \begin{pmatrix} S_{\widehat{D}}(\omega_{11}) & S_{\widehat{C}}(\omega_{21}) \\ S_{\widehat{B}}(\omega_{12}) & S_{\widehat{A}}(\omega_{22}) \end{pmatrix}.$$

We now construct a functional $\varepsilon_{\widehat{E}}$. Put

$$\varepsilon_{\widehat{B}} : \widehat{B} \to k : \omega \to \omega(1_B),$$

which makes sense since one can evaluate elements of $\widehat{B}$ on multipliers of B, using the specific form of the functionals in $\widehat{B}$. Put

$$\varepsilon_{\widehat{C}} : \widehat{C} \to k : \omega_{21} \to \omega_{21}(1_C).$$

Lemma 4.2.13. *The following identities hold:*

$$\varepsilon_{\widehat{B}}(\omega_{12} \cdot \omega_{22}) = \varepsilon_{\widehat{B}}(\omega_{12})\varepsilon_{\widehat{A}}(\omega_{22}),$$

$$\varepsilon_{\widehat{C}}(\omega_{22} \cdot \omega_{21}) = \varepsilon_{\widehat{A}}(\omega_{22})\varepsilon_{\widehat{C}}(\omega_{21}),$$

$$\varepsilon_{\widehat{A}}(\omega_{21} \cdot \omega_{12}) = \varepsilon_{\widehat{C}}(\omega_{21})\varepsilon_{\widehat{B}}(\omega_{12}).$$

Proof. This is immediately verified, using that the compositions inside $\widehat{E}$ which are used are duals of the maps γ_C, α_B and β_A, all of which are unital (when extended to the respective multiplier algebra).

□

By the previous lemma and Corollary 4.2.10, we can define a homomorphism

$$\varepsilon_{\widehat{D}} : \widehat{D} \to k : \omega_{12} \cdot \omega_{21} \to \varepsilon_{\widehat{B}}(\omega_{12})\varepsilon_{\widehat{C}}(\omega_{21}).$$

We can then also collect these ε's into the single map

$$\varepsilon_{\widehat{E}} : \widehat{E} \to k : \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix} \to \varepsilon_{\widehat{D}}(\omega_{11}) + \varepsilon_{\widehat{C}}(\omega_{21}) + \varepsilon_{\widehat{B}}(\omega_{12}) + \varepsilon_{\widehat{A}}(\omega_{22}).$$

We now gradually build a comultiplication on $\widehat{E}$.

Let $(B \odot B)^*$ be the dual of the vector space $B \odot B$. We can endow $(B \odot B)^*$ with two right $\widehat{A}$-module structures: for $\omega \in (B \odot B)^*$ and $\omega_{22} \in \widehat{A}$, we define

$$(\omega \cdot (1_{\widehat{A}} \otimes \omega_{22}))(b \otimes b') := \omega(b \otimes (\omega_{22} \cdot b')),$$

$$(\omega \cdot (\omega_{22} \otimes 1_{\widehat{A}}))(b \otimes b') := \omega((\omega_{22} \cdot b) \otimes b').$$

Note that we can embed $\widehat{B} \odot \widehat{B}$ inside $(B \odot B)^*$ in the natural way, and that this embedding respects both right $\widehat{A}$-module structures (which on $\widehat{B} \odot \widehat{B}$ are just the right module structures by multiplication with $\widehat{A}$ on the right on either the second or first leg).

Similarly, we can let $\widehat{C}$ act on the left of $(B \odot B)^*$ (either on 'the first or second leg'), obtaining then elements of either $(A \odot B)^*$ or $(B \odot A)^*$. First note that we have natural B-valued pairing

$$A \times \widehat{C} \to B : (a, \omega_{21}) \to a \cdot \omega_{21} := \omega_{21}(a_{[1]})a_{[2]}.$$

Then we define the mentioned left action as

$$((\omega_{21} \otimes 1_{\widehat{D}}) \cdot \omega)(a \otimes b) = \omega((a \cdot \omega_{21}) \otimes b),$$

$$((1_{\widehat{D}} \otimes \omega) \cdot \omega_{12})(b \otimes a) = \omega(b \otimes (a \cdot \omega_{21})),$$

for $\omega \in (B \odot B)^*$.

Definition 4.2.14. *Let B be a right A-Galois object. The* comultiplication $\Delta_{\widehat{B}}$ on $\widehat{B}$ *is the restriction of the transpose $M_B^t : B^* \to (B \odot B)^*$ to the space $\widehat{B}$.*

We then denote

$$\Delta_{\widehat{B}}(\omega_{12}) = \omega_{12(1)} \otimes \omega_{12(2)}$$

for $\omega_{12} \in \widehat{B}$, using the same purely formal Sweedler notation as for multiplier Hopf algebras. If $\omega_{22} \in \widehat{A}$, we then also denote

$$\Delta_{\widehat{B}}(\omega_{12}) \cdot (1_{\widehat{A}} \otimes \omega_{22}) =: \omega_{12(1)} \otimes (\omega_{12(2)} \cdot \omega_{22}),$$

and similarly for the other leg and for actions of $\widehat{C}$ on the left. We note then that $\widehat{B}$ with this comultiplication will be an instance (and in fact the most general instance) of a comonoidal Morita $\widehat{A}$-module, briefly mentioned at the end of the previous section.

Lemma 4.2.15. *The maps*

$$\widehat{B} \odot \widehat{A} \to (B \odot B)^* : \omega_{12} \otimes \omega_{22} \to \Delta_{\widehat{B}}(\omega_{12}) \cdot (1_{\widehat{A}} \otimes \omega_{22}),$$

$$\widehat{B} \odot \widehat{A} \to (B \odot B)^* : \omega_{12} \otimes \omega_{22} \to \Delta_{\widehat{B}}(\omega_{12}) \cdot (\omega_{22} \otimes 1_{\widehat{A}})$$

induce bijections $\hat{B} \odot \hat{A} \rightarrow \hat{B} \odot \hat{B}$. *Also, the maps*

$$\hat{C} \odot \hat{B} \rightarrow (A \odot B)^* : \omega_{21} \otimes \omega_{12} \rightarrow (\omega_{21} \otimes 1_{\hat{D}}) \cdot \Delta_{\hat{B}}(\omega_{12}),$$

$$\hat{C} \odot \hat{B} \rightarrow (B \odot A)^* : \omega_{21} \otimes \omega_{12} \rightarrow (1_{\hat{D}} \otimes \omega_{21}) \cdot \Delta_{\hat{B}}(\omega_{12})$$

induce bijections $\hat{C} \odot \hat{B} \rightarrow \hat{A} \odot \hat{B}$, *resp.* $\hat{C} \otimes \hat{B} \rightarrow \hat{B} \otimes \hat{A}$.

Proof. Note that

$$(\Delta_{\hat{B}}(\omega_{12}) \cdot (1_{\hat{A}} \otimes \omega_{22}))(b \otimes b') = (\omega_{12} \otimes \omega_{22})(G(b \otimes b')),$$

with G the Galois map for α_B. Hence the first map in the Lemma coincides with the restriction of the transpose G^t of G, and hence is injective, by the surjectivity of G. An easy calculation further shows that

$$G^t((\varphi_B(\,\cdot\, b) \otimes \varphi_A(a\,\cdot\,)) = \varphi_B(\,\cdot\, a^{[1]}b) \otimes \varphi_B(a^{[2]}\,\cdot\,)$$

for $b \in B, a \in A$, by using Proposition 3.4.1. i). This shows that the range of the first map in the lemma is exactly $\hat{B} \odot \hat{A}$, by Corollary 3.5.2.

The bijectivity statement concerning the second map follows in a similar fashion, using that also

$$B \odot A \rightarrow B \odot B : b \otimes b' \rightarrow \alpha_B(b)(b' \otimes 1)$$

is a bijection.

For the third map, note that

$$(\omega_{21} \cdot \omega_{12(1)}) \otimes \omega_{12(2)})(a \otimes b) = (\omega_{21} \otimes \omega_{12})(a_{[1]} \otimes a_{[2]}b).$$

Since the map

$$A \odot B \rightarrow C \odot B : a \otimes b \rightarrow a_{[1]} \otimes a_{[2]}b$$

is also bijective (by Proposition 3.1.2), the third map is injective. It is easy to calculate, using Proposition 3.4.1ii), that when $\omega_{21} = S_{\hat{B}}(\varphi_B(b\,\cdot\,))$ and $\omega_{12} = \varphi_B(b'\,\cdot\,)$, then the third map sends

$$\omega_{21} \otimes \omega_{12} \rightarrow (S_{\hat{A}}(\varphi_A(b_{(1)}\,\cdot\,))) \otimes \varphi_B(b'b_{(0)}\,\cdot\,),$$

which proves the bijectivity associated with the third map (by bijectivity of the Galois map). The proof of the bijectivity of the fourth map is entirely similar.

□

By the previous lemma, and using further the unitality of $\widehat{C}$ as a left $\widehat{A}$-module and the fact that $\widehat{D} = \widehat{B} \cdot \widehat{C}$, we can (and will) regard $\Delta_{\widehat{B}}(\omega_{12})$ as an element of $M_{1;2}(\widehat{B} \odot \widehat{B})$.

We then also define

$$\Delta_{\widehat{C}} : \widehat{C} \to M_{1;2}(\widehat{C} \odot \widehat{C}) : S_{\widehat{B}}(\omega_{12}) \to S_{\widehat{B}}(\omega_{12(2)}) \otimes S_{\widehat{B}}(\omega_{12(1)}).$$

We may also interpret $\Delta_{\widehat{C}}(\omega_{21})$ as the functional

$$c \otimes c' \to \omega_{21}(cc')$$

on $\widehat{C} \odot \widehat{C}$.

Lemma 4.2.16. *The following identities hold:*

$$\Delta_{\widehat{B}}(\omega_{12} \cdot \omega_{22}) = \Delta_{\widehat{B}}(\omega_{12}) \cdot \Delta_{\widehat{A}}(\omega_{22}),$$
$$\Delta_{\widehat{C}}(\omega_{22} \cdot \omega_{21}) = \Delta_{\widehat{A}}(\omega_{22}) \cdot \Delta_{\widehat{C}}(\omega_{21}),$$
$$\Delta_{\widehat{A}}(\omega_{21} \cdot \omega_{12}) = \Delta_{\widehat{C}}(\omega_{21}) \cdot \Delta_{\widehat{B}}(\omega_{12}),$$

where the multiplications are inside $M(\widehat{E} \odot \widehat{E})$.

Proof. Take $\omega_{21} \in \widehat{C}$. Then both $(\omega_{21} \otimes 1_{\widehat{D}}) \cdot (\Delta_{\widehat{B}}(\omega_{12} \cdot \omega_{22}))$ and $(\omega_{21} \otimes 1_{\widehat{D}}) \cdot \Delta_{\widehat{B}}(\omega_{12}) \cdot \Delta_{\widehat{A}}(\omega_{22})$ are inside $\widehat{A} \odot \widehat{B}$, and for the first identity, it is enough to check if these are equal. For $a \in A$ and $b \in B$, we compute:

$$\begin{aligned}
&((\omega_{21} \otimes 1_{\widehat{D}})\Delta_{\widehat{B}}(\omega_{12})\Delta_{\widehat{A}}(\omega_{22}))(a \otimes b) \\
&= ((\omega_{21} \cdot \omega_{12(1)}) \cdot \omega_{22(1)} \otimes \omega_{12(2)} \cdot \omega_{22(2)})(a \otimes b) \\
&= (\omega_{21} \cdot \omega_{12(1)})(a_{(1)})\omega_{22(1)}(a_{(2)})(\omega_{12(2)} \cdot \omega_{22(2)})(b) \\
&= (\omega_{21} \cdot \omega_{12(1)})(a_{(1)})(\omega_{12(2)} \cdot \omega_{22}(a_{(2)} \cdot))(b) \\
&= \omega_{21}(a_{(1)[1]})\omega_{12(1)}(a_{(1)[2]})\omega_{12(2)}(b_{(0)})\omega_{22}(a_{(2)}b_{(1)}) \\
&= \omega_{21}(a_{(1)[1]})\omega_{12}(a_{(1)[2]}b_{(0)})\omega_{22}(a_{(2)}b_{(1)}),
\end{aligned}$$

where the reader should check for himself that at every stage, the expressions are well-covered. On the other hand,

$$\begin{aligned}
&(\omega_{21} \otimes 1_{\widehat{D}}) \cdot (\Delta_{\widehat{B}}(\omega_{12} \cdot \omega_{22}))(a \otimes b) \\
&= \omega_{21}(a_{[1]})(\omega_{12} \cdot \omega_{22})(a_{[2]}b) \\
&= \omega_{21}(a_{[1]})\omega_{12}((a_{[2]}b)_{(0)})\omega_{22}((a_{[2]}b)_{(1)}) \\
&= \omega_{21}(a_{[1]})\omega_{12}((a_{[2](0)}b_{(0)})\omega_{22})(a_{[2](1)}b_{(1)}).
\end{aligned}$$

This then equals the previous expression by Proposition 3.7.2.

The second identity and third identity are proven in an entirely similar way, reducing each time to the coassociativity statements in Proposition 3.7.2.
□

Hence, by Corollary 4.2.10 and Lemma 2.2.6, we get a well-defined homomorphism

$$\Delta_{\widehat{D}} : \widehat{D} \to M(\widehat{D} \odot \widehat{D}) : \omega_{12} \cdot \omega_{21} \to \Delta_{\widehat{B}}(\omega_{12}) \cdot \Delta_{\widehat{C}}(\omega_{21}).$$

We can then also combine the Δ's into a homomorphism

$$\Delta_{\widehat{E}} : \widehat{E} \to M(\widehat{E} \odot \widehat{E}),$$

by taking their direct sum.

Proposition 4.2.17. *Let B be a right A-Galois object. Then the triple* $(\widehat{E}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \Delta_{\widehat{E}})$ *is a linking multiplier weak Hopf algebra, with $\varepsilon_{\widehat{E}}$ as its counit and $S_{\widehat{E}}$ as its antipode.*

Proof. We first show that $\Delta_{\widehat{B}}$ is coassociative. For

$$\begin{aligned}
&((\iota_{\widehat{A}} \otimes \Delta_{\widehat{B}})((\omega_{21} \otimes 1_{\widehat{D}})\Delta_{\widehat{B}}(\omega_{12})) \cdot (1_{\widehat{A}} \otimes 1_{\widehat{A}} \otimes \omega_{22}))(a \otimes b \otimes b') \\
&\quad = ((\omega_{21} \otimes 1_{\widehat{D}})\Delta_{\widehat{B}}(\omega_{12}))(a \otimes bb'_{(0)})\omega_{22}(b'_{(1)}) \\
&\quad = \omega_{21}(a_{[1]})\omega_{12}(a_{[2]}(bb'_{(0)}))\omega_{22}(b'_{(1)}) \\
&\quad = \omega_{21}(a_{[1]})\omega_{12}((a_{[2]}b)b'_{(0)})\omega_{22}(b'_{(1)}) \\
&\quad = \ldots \\
&\quad = ((\omega_{21} \otimes 1_{\widehat{D}} \otimes 1_{\widehat{D}}) \cdot (\Delta_{\widehat{B}} \otimes \iota_{\widehat{B}})(\Delta_{\widehat{B}}(\omega_{12})(1_{\widehat{A}} \otimes \omega_{22})))(a \otimes b \otimes b'),
\end{aligned}$$

which is easily seen to be sufficient to conclude that

$$(\iota_{\widehat{B}} \otimes \Delta_{\widehat{B}})\Delta_{\widehat{B}}(\omega_{12}) = (\Delta_{\widehat{B}} \otimes \iota_{\widehat{B}})\Delta_{\widehat{B}}(\omega_{12}) \in M(\widehat{E} \odot \widehat{E} \odot \widehat{E}).$$

Then $\Delta_{\widehat{C}}$ is coassociative by an entirely similar argument, and $\Delta_{\widehat{D}}$ is coassociative by definition (and a small further argument). All of this combined shows that $\Delta_{\widehat{E}}$ is coassociative.

Now we have to check the bijectivity of a certain family of maps which are given in Definition 4.1.1. We only check the bijectivity of the maps which apply a comultiplication to the first leg, and then multiply to the right with the second leg (i.e., those in the first group of eight morphisms in Definition 4.1.1). Now the bijectivity of those morphisms involving only $\widehat{C}, \widehat{B}$ and $\widehat{A}$ follow by entirely similar methods as (or some even directly by) in Lemma 4.2.15. Only three maps remain then. For example, we have to show that

$$\widehat{C} \odot \widehat{D} \to \widehat{C} \odot \widehat{C} : \omega_{21} \otimes \omega_{11} \to \Delta_{\widehat{C}}(\omega_{21}) \cdot (1_{\widehat{D}} \otimes \omega_{11})$$

is bijective. But

$$\begin{aligned} \Delta_{\widehat{C}}(\widehat{C})(1_{\widehat{D}} \otimes \widehat{D}) &= (\Delta_{\widehat{C}}(\widehat{C})(1_{\widehat{D}} \otimes \widehat{B}))(1_{\widehat{D}} \otimes \widehat{C}) \\ &= (\widehat{C} \otimes \widehat{B})(1_{\widehat{D}} \otimes \widehat{C}) \\ &= \widehat{C} \otimes \widehat{D}, \end{aligned}$$

proving surjectivity. On the other hand, if

$$\sum_i \Delta_{\widehat{C}}(\omega_{21}^i) \cdot (1_{\widehat{D}} \otimes \omega_{11}^i) = 0,$$

then

$$\sum_i \Delta_{\widehat{C}}(\omega_{21}^i) \cdot (1_{\widehat{D}} \otimes (\omega_{11}^i \cdot \omega_{12})) = 0,$$

for all $\omega_{12} \in \widehat{B}$. Hence

$$\sum_i \omega_{21}^i \otimes (\omega_{11}^i \cdot \omega_{12}) = 0$$

for all $\omega_{12} \in \widehat{B}$, which implies

$$\sum_i \omega_{21}^i \otimes \omega_{11}^i = 0,$$

proving injectivity. In an entirely similar fashion, the two remaining maps can be shown to be bijective.

Finally, it is trivial to see that $\varepsilon_{\widehat{E}}$ will be the counit for this linking multiplier weak Hopf algebra. Also the proof that $S_{\widehat{E}}$ is the antipode is straightforward enough to safely omit the proof.

□

Now we construct on the multiplier Hopf algebra $\widehat{D}$ a non-zero left invariant functional, showing that it is in fact an algebraic quantum group. Again, this is a non-trivial procedure, as even for unital linking algebras, there is for example in general no canonical way to transport some functional on one corner to a functional on the other corner. We use the method of proof from [23], which is less cumbersome than the original construction of [19].

Definition-Proposition 4.2.18. *Let B be a right A-Galois object. Denote by $\sigma_{\widehat{B}} : \widehat{B} \to B^*$ the map such that*

$$((\sigma_{\widehat{B}})(\omega_{12}))(b) = \omega_{12}(S_B^2(b)\delta_B^{-1}).$$

Then $\sigma_{\widehat{B}}$ has range in $\widehat{B}$. We call it the modular automorphism of $\widehat{B}$ with respect to $\varphi_{\widehat{A}}$.

Proof. This is easily verified: if $\omega_{12} = \varphi_B(\,\cdot\, b)$, then

$$\begin{aligned}(\sigma_{\widehat{B}}(\omega_{12}))(b') &= \varphi_B(S_B^2(b')\cdot\delta_B^{-1}b) \\ &= \nu_A\varphi_B(b'\delta_B^{-1}S_B^{-2}(b)),\end{aligned}$$

using the relative invariance of φ_B, and the invariance of δ_B with respect to S_B^2. □

Proposition 4.2.19. *The functional*

$$\varphi_{\widehat{D}} : \widehat{D} \to k : \omega_{12}\cdot\omega_{21} \to \varphi_{\widehat{A}}(\omega_{21}\cdot\sigma_{\widehat{B}}(\omega_{12}))$$

is well-defined, and determines a non-zero left invariant functional on $\widehat{D}$.

Proof. We first verify that

$$\sigma_{\widehat{B}}(\omega_{12}\cdot\omega_{22}) = \sigma_{\widehat{B}}(\omega_{12})\cdot\sigma_{\widehat{A}}(\omega_{22}).$$

For $b \in A$, we have

$$\begin{aligned}(\sigma_{\widehat{B}}(\omega_{12}\cdot\omega_{22}))(b) &= (\omega_{12}\otimes\omega_{22})(\alpha_B(S_B^2(b)\delta_B^{-1})) \\ &= (\omega_{12}\otimes\omega_{22})((S_B^2(b_{(0)})\delta_B^{-1})\otimes(S_A^2(b_{(1)})\delta_A^{-1})) \\ &= (\sigma_{\widehat{B}}(\omega_{12})\cdot\sigma_{\widehat{A}}(\omega_{22}))(b),\end{aligned}$$

using the appropriate identities from the previous chapter. Then from this, we conclude that

$$\begin{aligned}\varphi_{\widehat{D}}((\omega_{12}\cdot\omega_{22})\cdot\omega_{12}) &= \varphi_{\widehat{A}}(\omega_{21}\cdot\sigma_{\widehat{B}}(\omega_{12})\cdot\sigma_{\widehat{A}}(\omega_{22})) \\ &= \varphi_{\widehat{A}}(\omega_{22}\cdot\omega_{21}\cdot\sigma_{\widehat{B}}(\omega_{12})) \\ &= \varphi_{\widehat{D}}(\omega_{12}\cdot(\omega_{22}\cdot\omega_{12})),\end{aligned}$$

which by Corollary 4.2.10 shows that $\varphi_{\widehat{D}}$ is well-defined.

We now show that $\varphi_{\widehat{D}}$ is left invariant. We first prove another identity, namely

$$\Delta_{\widehat{B}}(\sigma_{\widehat{B}}(\omega_{12})) = (S^2_{\widehat{B}} \otimes \sigma_{\widehat{B}})\Delta_{\widehat{B}}(\omega_{12}).$$

Again, this is straightforward to verify:

$$\begin{aligned}
\Delta_{\widehat{B}}(\sigma_{\widehat{B}}(\omega_{12}))(b \otimes b') &= \omega_{12}(S^2_B(bb')\delta_B^{-1}) \\
&= \omega_{12}(S^2_B(b)S^2_B(b')\delta_B^{-1}) \\
&= \Delta_{\widehat{B}}(\omega_{12})(S^2_B(b) \otimes S^2_B(b')\delta_B^{-1}) \\
&= ((S^2_{\widehat{B}} \otimes \sigma_{\widehat{B}})\Delta_{\widehat{B}}(\omega_{12}))(b \otimes b').
\end{aligned}$$

Then we compute for $\omega_{11} = \omega_{12} \cdot \omega_{21} \in \widehat{D}$ that

$$\begin{aligned}
&\varphi_{\widehat{D}}(\omega_{11(2)})\omega'_{21}\omega_{11(1)}\omega'_{12} \\
&= \varphi_{\widehat{A}}(\omega_{21(2)}\sigma_{\widehat{B}}(\omega_{12(2)}))\omega'_{21}\omega_{12(1)}\omega_{21(1)}\omega'_{12} \\
&= \varphi_{\widehat{A}}(\omega_{21(2)}(\sigma_{\widehat{B}}(\omega_{12}))_{(2)})S^{-2}_{\widehat{B}}(S^2_{\widehat{B}}(\omega'_{21}) \cdot (\sigma_{\widehat{B}}(\omega_{12}))_{(1)})\omega_{21(1)}\omega'_{12} \\
&= \varphi_{\widehat{A}}(\omega_{21(4)}(\sigma_{\widehat{B}}(\omega_{12}))_{(2)})S^{-2}_{\widehat{B}}(S^2_{\widehat{B}}(\omega'_{21}) \\
&\qquad \cdot S_{\widehat{C}}(\omega_{21(2)})\omega_{21(3)}(\sigma_{\widehat{B}}(\omega_{12}))_{(1)})\omega_{21(1)}\omega'_{12} \\
&= \varphi_{\widehat{A}}((\omega_{21(3)}(\sigma_{\widehat{B}}(\omega_{12})))_{(2)})S^{-2}_{\widehat{B}}(S^2_{\widehat{B}}(\omega'_{21}) \\
&\qquad \cdot S_{\widehat{C}}(\omega_{21(2)})(\omega_{21(3)}(\sigma_{\widehat{B}}(\omega_{12})))_{(2)})\omega_{21(1)}\omega'_{12} \\
&= \varphi_{\widehat{A}}(\omega_{21(3)}\sigma_{\widehat{B}}(\omega_{12}))S^{-2}_{\widehat{B}}(S^2_{\widehat{B}}(\omega'_{21}) \cdot S_{\widehat{C}}(\omega_{21(2)}))\omega_{21(1)}\omega'_{12} \\
&= \varphi_{\widehat{A}}(\omega_{21(3)}\sigma_{\widehat{B}}(\omega_{12}))\omega'_{21}S^{-1}_{\widehat{C}}(\omega_{21(2)})\omega_{21(1)}\omega'_{12} \\
&= \varphi_{\widehat{A}}(\omega_{21}\sigma_{\widehat{B}}(\omega_{12}))\omega'_{21}\omega'_{12} \\
&= \varphi_{\widehat{D}}(\omega_{12} \cdot \omega_{21})\omega'_{21}\omega'_{12} \\
&= \varphi_{\widehat{D}}(\omega_{11})\omega'_{12} \cdot \omega'_{21},
\end{aligned}$$

where we have twice used the antipode property for $S_{\widehat{C}}$. This proves that $\varphi_{\widehat{D}}$ is a left invariant functional.

□

Corollary 4.2.20. *Let B be a right A-Galois object. Then the associated linking multiplier weak Hopf algebra $(\widehat{E}, e)$ is a linking algebraic quantum groupoid.*

4.3 Bi-Galois objects from linking algebraic quantum groupoids

In this section, we construct from the datum of a linking algebraic quantum group between a bi-Galois object. In fact, this is done by a duality argument which is completely similar to the construction of the dual of an algebraic quantum group in [93]. Therefore, we will be rather brief, and not provide all proofs.

Let $(\widehat{E}, e)$ be a linking algebraic quantum groupoid (which we write as a dual already to have compatibility with previous notations). Then on $\widehat{E}$ we can construct the functionals

$$\varphi_{\widehat{E}} : \widehat{E} \to k : \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix} \to \varphi_{\widehat{D}}(\omega_{11}) + \varphi_{\widehat{A}}(\omega_{22})$$

and

$$\psi_{\widehat{E}} : \widehat{E} \to k : \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix} \to \psi_{\widehat{D}}(\omega_{11}) + \psi_{\widehat{A}}(\omega_{22}).$$

The same techniques as used in the third section of [92] or the previous chapter, will let us conclude that $\varphi_{\widehat{E}}$ and $\psi_{\widehat{E}}$ possess modular automorphisms (see also [23]). In fact, these will then split up into bijections $\widehat{E}_{ij} \to \widehat{E}_{ij}$, for which we then continue to use the obvious notation.

More trivially, $\varphi_{\widehat{E}}$ and $\psi_{\widehat{E}}$ are linked by a modular element: if we define $\delta_{\widehat{E}} := \begin{pmatrix} \delta_{\widehat{D}} & 0 \\ 0 & \delta_{\widehat{A}} \end{pmatrix}$, then $\varphi_{\widehat{E}}(\,\cdot\,\delta_{\widehat{E}}) = \psi_{\widehat{E}}$.
Then define E to be the restricted dual of $\widehat{E}$ w.r.t. $\varphi_{\widehat{E}}$ or $\psi_{\widehat{E}}$, i.e. the space of functionals of the form $\varphi_{\widehat{E}}(\,\cdot\,\omega)$ with $\omega \in \widehat{E}$ (where it doesn't matter which invariant functional we choose, or where we put the element inside). As in the fourth section of [93], we will obtain that dual to the comultiplication on $\widehat{E}$, there exists a non-degenerate algebra structure on E, and that dual to the multiplication on $\widehat{E}$, there will exist a u.e. coassociative homomorphism $\Delta_E \to M_{1;2}(E \odot E)$.

Now it is further easily checked that E will be a direct sum algebra of non-degenerate algebras E_{11}, E_{12}, E_{21} and E_{22}, where $E_{ij} = \{\varphi_{\widehat{E}_{ii}}(\,\cdot\,\omega_{ji}) \mid \omega_{ji} \in \widehat{E}_{ji}\}$, and that, denoting by $p_{ij} = 1_{E_{ij}} \in M(E)$, we have $\Delta_E(p_{ij}) = (p_{i1} \otimes p_{1j}) + (p_{i2} \otimes p_{2j})$. Then as for co-linking weak Hopf algebras, Δ_E

splits into u.u.e. homomorphisms $\Delta_{ij}^k : E_{ij} \to M_{1;2}(E_{ik} \odot E_{kj})$, and we can identify (E_{11}, Δ_{11}^1) with (D, Δ_D), and (E_{22}, Δ_{22}^2) with (A, Δ_A). We then also use the further notation as introduced for co-linking weak Hopf algebras after Definition 1.3.7. We easily find that $\alpha_B : B \to M_{1;1}(B \odot A)$ is then a (reduced) right coaction, and it will make B into an A-Galois object, for example by observing that by definition of the multiplication in B and the coaction of A on B, the Galois map G for α_B is the dual (i.e, the restricted transpose) of the map

$$\widehat{B} \odot \widehat{A} \to \widehat{B} \odot \widehat{B} : \omega_{12} \otimes \omega_{22} \to \Delta_{\widehat{B}}(\omega_{12})(\omega_{22} \otimes 1),$$

which we know to be bijective by Proposition 4.1.2. (We used here of course that the transpose of the inverse of this last map again sends $B \odot A$ into $B \odot B$, but this is also something which is straightforward to establish.)

Similarly, the map $\gamma_B : B \to M_{1;2}(D \odot B)$ turns B into a left D-Galois object, and since it is easily verified that γ_B and α_B commute, we have constructed from $\widehat{E}$ a bi-Galois object. Furthermore, if $(\widehat{E}, e)$ was the linking algebraic quantum groupoid *constructed* from a right A-Galois object B, then it is straightforward to verify that (B, α_B) coincides with the Galois object as constructed from $(\widehat{E}, e)$. So combining the construction of a bi-Galois object from a linking algebraic quantum group together with the construction of a linking algebraic quantum groupoid from a right Galois object, we have proven half of the following Proposition.

Proposition 4.3.1. *Let A be an algebraic quantum group, and B a right A-Galois object. Then there exists an algebraic quantum group D and coaction γ_B of D on B making (B, γ_B, α_B) into a bi-Galois object. Moreover, if D_1 is another algebraic quantum group, and γ_B^1 a coaction of D_1 on B, making $(B, \gamma_B^1, \alpha_B)$ into a Galois object, then there exists an isomorphism*

$$\Phi_{D^1} : D^1 \to D$$

such that

$$(\Phi_D \otimes \iota_B)\gamma_B^1 = \gamma_B.$$

We will not give a full proof of the uniqueness statement. Suffice it to say that one can also construct directly from a bi-Galois object a linking algebraic quantum groupoid, where now the multiplications between the $\widehat{B}$ and $\widehat{C}$ are not considered by composition of linear maps on some vector space, but directly by dualizing the external comultiplications of A and D. Then by using Proposition 4.1.4, we see that necessarily our newly constructed linking quantum groupoid must be isomorphic to the one constructed solely from the right A-Galois object (B, α_B).

We end with a generalization of a formula, known as *Radford's formula*, which is well-known for Hopf algebras with integrals (and more generally, for algebraic quantum groups). It gives a formula for the fourth power of the antipode in terms of the modular elements.

Proposition 4.3.2. *Let B be a right Galois object, and $(\widehat{E}, e)$ its associated linking algebraic quantum groupoid. Then*

$$S_B^4(b) = (\delta_{\widehat{A}} \cdot (\delta_B^{-1} b \delta_B) \cdot \delta_{\widehat{D}}^{-1})$$

for $b \in B$.

Proof. Recall that $S_B^2(b) = \delta_{\widehat{A}} \cdot \sigma_B(b)$ for $b \in B$, by definition of S_B^2. Now since S_B^2 is uniquely determined as the transpose of the dual of the antipode on $\widehat{E}$, which is unique, we should also have a similar formula for S_B^2 starting from the *left* D-Galois object (B, γ_B). By analogy with the case of algebraic quantum groups, we see that this formula must be

$$S_B^2(b) = ((\sigma_B)^{-1}(b)) \cdot \delta_{\widehat{D}}^{-1},$$

since ψ_B must be the $\delta_{\widehat{D}}^{-1}$-invariant functional for γ_B by a uniqueness argument. Since $\sigma_B(b) = \delta_B \sigma_B(b) \delta_B^{-1}$ for $b \in B$, and since B is a $\widehat{A}$-$\widehat{D}$-bimodule, combining our formulas leads to

$$S_B^4(b) = (\delta_{\widehat{A}} \cdot (\delta_B^{-1} b \delta_B) \cdot \delta_{\widehat{D}}^{-1})$$

for all $b \in B$.

□

4.4 Concerning *-structures

Suppose now again that A is a *-algebraic quantum group, and B a right *-Galois object. We first put a *-structure on the associated linking algebraic quantum groupoid $(\widehat{E}, e)$, making it into a linking multiplier weak Hopf *-algebra. We will be rather brief in our discussion, leaving easy verifications to the reader.

For $\omega_{12} \in \widehat{B}$, define $\omega_{12}^* \in C^*$ as the functional

$$\omega_{12}^*(c) := \overline{\omega_{12}(S_C(c)^*)}.$$

Then we will have $\omega_{12}^* \in \widehat{C}$, the *-operation then becoming bijective. We then define a *-operation from $\widehat{C}$ to $\widehat{B}$ simply as the inverse of the one from $\widehat{B}$ to $\widehat{C}$. Then $(\omega_{21} \cdot \omega_{12})^* = \omega_{12}^* \cdot \omega_{21}^*$, and $(\omega_{12} \cdot \omega_{22})^* = \omega_{22}^* \cdot \omega_{12}^*$ and $(\omega_{22} \cdot \omega_{21})^* = \omega_{21}^* \cdot \omega_{22}^*$. By Corollary 4.2.10, we conclude that there is a well-defined *-operation on $\widehat{D}$ by putting

$$(\omega_{12} \cdot \omega_{21})^* := \omega_{21}^* \cdot \omega_{12}^*.$$

Then the direct sum of the *-operations makes $\widehat{E}$ into a *-algebra, and moreover, $\Delta_{\widehat{E}}$ is *-preserving. Hence $(\widehat{E}, e)$ is a linking multiplier weak Hopf *-algebra.

We now make the dual bi-Galois object (B, γ_B, α_B) into a bi-*-Galois object (and in particular, make D into a multiplier Hopf *-algebra). Define a *-operation on D by the following dualization process:

$$\omega_{11}(d^*) := \overline{(S_{\widehat{D}}(\omega_{11})^*)(d)}.$$

Then it is straightforward to see that D becomes a *-algebra, and that Δ_D and γ_B will be *-preserving. Hence (B, γ_B, α_B) will be a A-D-bi-*-Galois object.

We show now that the property of being a *-algebraic quantum group is preserved under reflection along a Galois object, i.e., that the above constructed multiplier Hopf *-algebra D is in fact a *-algebraic quantum group.

We will show this by constructing a positive right invariant functional ψ_D on D. In fact, let ψ_B be a positive α_B-invariant functional on B (see Corollary 3.9.5). Then, since for any right invariant functional ψ_D on D, we have that $(\psi_D \otimes \iota)\gamma_B$ produces an α_B-invariant functional on B, we can, by the 'uniqueness' of an α_B-invariant functional on B, choose ψ_D in such a way that $(\psi_D \otimes \iota)\gamma_B$ coincides with ψ_B. Now take $d \in D$ and $b \in B$ with $\varphi_B(b^*b) = 1$. Write $d \otimes b$ as $\sum_i \gamma_B(b_i)(1 \otimes b_i')$, then $d^*d = \sum_{i,j}(\iota_D \otimes \varphi_B)((1_D \otimes (b_i')^*)\gamma_B((b_i)^*b_j)(1_D \otimes b_j'))$. Applying ψ_D, we get

$$\begin{aligned} \psi_D(d^*d) &= \sum_{i,j} \varphi_B((b_i')^*b_j')\psi_B(b_i^*b_j) \\ &\geqslant 0, \end{aligned}$$

since the matrices $(a_{i,j}) = (\varphi_B((b'_i)^* b'_j))$ and $(b_{i,j}) = (\psi_B((b_i)^* b_j))$ are both positive definite. Hence

Theorem 4.4.1. *If B is a right *-Galois object for a *-algebraic quantum group A, then the reflected algebraic quantum group (D, Δ_D) also has the structure of a *-algebraic quantum group.*

We can then also complete the discussion in section 3.9 concerning diagonalizability. For let B be a right *-Galois object. Then by Radford's formula, Proposition 4.3.2, S^4 is a composition of left and right multiplication with δ_B and its inverse, and left and right multiplication with $\delta_{\widehat{D}}$ or $\delta_{\widehat{A}}$ and their inverses. Since $B = \widehat{A} \cdot B \cdot \widehat{D}$, all these operations are diagonalizable, hence the same is true of S_B^4. Since S_B^2 is self-adjoint, also S_B^2 is diagonalizable, and then the same is true for σ_B, since $\sigma_B(b) = S_B^2(\delta_{\widehat{A}}^{-1} \cdot b)$ for all $b \in B$. Since S_B^2, σ_B and left and right multiplication with δ_B commute, they are all simultaneously diagonalizable. Finally, since φ_B is positive, one easily sees that σ_B must have positive eigenvalues, and then the same is also true of S_B^2.

4.5 An example

In this section, we present a family of examples of Galois objects for a certain class of algebraic quantum groups of compact type. While these last are of course special types of Hopf algebras, and thus could be treated solely in the framework of [71], we emphasize here the approach by duality (thus passing to algebraic quantum groups of discrete type). Another reason for including these examples is that the reflection along these Galois objects really produce a new algebraic quantum group of compact type. This is somewhat surprising, as the examples we present are infinite-dimensional generalizations of the Taft algebras, for which it is know that one always obtains an isomorphic copy of the original Hopf algebra when reflecting along a Galois object.

The mentioned class of algebraic quantum groups of compact type which we will use is the following. These examples can be found in [93] and [99], but we slightly generalize the construction to fit them both in one family.

Definition 4.5.1. *Let $n > 1$, $m \geqslant 1$ be natural numbers, and $\lambda \in k$ such that λ^m is a primitive n-th root of unity. Let $A_\lambda^{n,m}$ be the unital algebra over*

k generated by elements a, a^{-1} and b, and with defining relations: a^{-1} is the inverse of a, $ab = \lambda ba$ and $b^n = 0$. Then we can define a comultiplication on $A_\lambda^{n,m}$ determined on the generators by

$$\Delta(a) = a \otimes a,$$

$$\Delta(b) = b \otimes a^m + 1 \otimes b.$$

This makes $(A_\lambda^{n,m}, \Delta)$ an algebraic quantum group of compact type.

To prove that this comultiplication is indeed well-defined, we only have to use the well-known fact that $(s+t)^l = s^l + t^l$ when s, t are variables satisfying the commutation $st = qts$ with q a primitive l-th root of unity (see e.g. [52]). Now $(A_\lambda^{n,1}, \Delta)$ is the example in [99], and with the further relation $a^n = 1$, this reduces to the two-generator Taft algebras. The Hopf algebra $(A_\lambda^{n,2}, \Delta)$ is isomorphic with the example constructed in [93].

The left invariant functional φ of $(A, \Delta_A) = (A_\lambda^{n,m}, \Delta)$ is defined by

$$\varphi(a^p b^q) = \delta_{p,0}\delta_{q,n-1}, \qquad p \in \mathbb{Z}, 0 \leqslant q < n.$$

As A is infinite-dimensional, the dual $\widehat{A}$ is necessarily of discrete type and not compact, i.e. it is a genuine multiplier Hopf algebra. This is a difference with the Taft algebras, which are self-dual. Remark that there can still be defined a pairing between A and itself, but it will be degenerate.

In [62] the Galois objects for the Taft algebras were classified. It provides the motivation for the following construction. Fix $(A, \Delta_A) = (A_\lambda^{n,m}, \Delta)$ as above, and assume moreover that λ is a primitive n-th root of unity and m and n are coprime. The condition 'λ^m is a primitive n-th root of unity' follows from this assumption.

Definition 4.5.2. *Take $\mu \in k$. Let $B = B_{\lambda,\mu}^{n,m}$ be the unital algebra generated by x, x^{-1} and y, with the defining relations: x^{-1} is the inverse of x, $xy = \lambda yx$ and $y^n = \mu x^{mn}$. A right coaction α_B of A on B is defined on the generators by*

$$\alpha_B(x) = x \otimes a,$$

$$\alpha_B(y) = y \otimes a^m + 1 \otimes b.$$

It is straightforward to show that this has a well-defined extension to the whole of B, and that this is a coaction (since the coaction property only has to be checked on generators).

Proposition 4.5.3. *(B, α_B) is a right A-Galois object.*

Proof. First of all, we have to see if B is not trivial. We follow a standard procedure. Let V be a vector space over k which has a basis of vectors of the form $e_{p,q}$ with $p \in \mathbb{Z}$ and $0 \leqslant q < n$. Define operators x' and y' by

$$\begin{array}{lcrl} x' \cdot e_{p,q} & = & e_{p+1,q} & \text{for all } p \in \mathbb{Z}, 0 \leqslant q < n, \\ y' \cdot e_{p,q} & = & \lambda^{-p}\, e_{p,q+1} & \text{if } p \in \mathbb{Z}, 0 \leqslant q < n-1, \\ y' \cdot e_{p,n-1} & = & \mu\lambda^{-p}\, e_{p+nm,0} & \text{if } p \in \mathbb{Z}. \end{array}$$

Then it is easy to see that x' is invertible and that $x'y' = \lambda y'x'$. Also:

$$\begin{aligned} y'^n \cdot e_{p,q} &= \lambda^{-p(n-1-q)} y'^{1+q} \cdot e_{p,n-1} \\ &= \mu\lambda^{-p(n-1-q)}\lambda^{-p} y'^q \cdot e_{p+nm,0} \\ &= \mu\lambda^{-p(n-1-q)}\lambda^{-p}\lambda^{-pq} e_{p+nm,q} \\ &= \mu\lambda^{-pn} e_{p+nm,q} \\ &= \mu x'^{mn} \cdot e_{p,q}. \end{aligned}$$

This gives us a non-trivial representation of B. Moreover, it is easy to see that this representation is faithful.

Define by $\beta_A : A \to B^{\mathrm{op}} \otimes B$ the homomorphism generated by

$$\beta_A(a) = (x^{-1})^{\mathrm{op}} \otimes x,$$

$$\beta_A(b) = -(yx^{-m})^{\mathrm{op}} \otimes x^m + 1 \otimes y.$$

This is well-defined: for example, we have

$$\begin{aligned} \beta_A(b)^n &= ((-yx^{-m})^n)^{\mathrm{op}} \otimes x^{mn} + \mu(1 \otimes x^{mn}) \\ &= ((-1)^n \lambda^{mn(n-1)/2} + 1)\mu(1 \otimes x^{mn}) \\ &= 0, \end{aligned}$$

using that λ^m is a primitive root of unity. Denoting $\tilde{\beta}_A = (S_{B^{\mathrm{op}}} \otimes \iota)\beta_A$ with $S_{B^{\mathrm{op}}}$ the canonical map $B^{\mathrm{op}} \to B$, and writing $\tilde{\beta}_A(c) = c^{[1]} \otimes c^{[2]}$ for $c \in A$, it is easy to compute that

$$z_{(0)} z_{(1)}^{[1]} \otimes z_{(1)}^{[2]} = 1 \otimes z,$$

$$c^{[1]} c^{[2]}_{(0)} \otimes c^{[2]}_{(1)} = 1 \otimes c$$

for all $z \in \{x, y, x^{-1}\}$ and $c \in \{a, b, a^{-1}\}$, and hence for all $z \in B, c \in A$. This shows that the coaction α_B makes B into a Galois object. □

The extension $k \subseteq B$ will be cleft (see e.g. Definition 2.2.3. in [76]), by the comodule isomorphism $\Psi_B : B \to A : x^p y^q \to a^p b^q$, $p \in \mathbb{Z}$ and $0 \leqslant q < n$. The associated scalar-valued 2-cocycle ω is given by $\omega(a^p b^q \otimes a^r b^s) = 0$, except for $q = s = 0$, where it is 1, and when $q + s = n$, in which case it equals $\mu\lambda^{-rq}$. (We were pointed out by the referee of [19] that our Hopf algebra is... pointed, so that any Galois object is automatically cleft (see [45]).)

We determine the extra structure occurring in this example. First note that we have shown that the elements of the form $x^p y^q$ with $p \in \mathbb{Z}$ and $0 \leqslant q < n$ form a basis. Then we have

$$\begin{array}{lll} \varphi_B(x^p y^q) & = & \delta_{q,n-1}\delta_{p,0} \qquad\qquad \text{for} \quad p \in \mathbb{Z}, 0 \leqslant q < n, \\ \psi_B(x^p y^q) & = & \delta_{q,n-1}\delta_{p,m(1-n)}\lambda^{-m} \qquad \text{for} \quad p \in \mathbb{Z}, 0 \leqslant q < n, \end{array}$$

$$\delta_B = x^{(n-1)m}$$

$$\begin{array}{llllll} \sigma_B(x) & = & \lambda^{-1}x, & S_B^2(x) & = & x \\ S_B^2(y) & = & y, & \theta_B(y) & = & \lambda^m y, \end{array}$$

by some easy computations (where we have used notation as in the previous chapter). Of course, the fact that the integrals on B are 'the same' as the ones A (after applying Ψ_B) is immediate from the fact that A and B coincide as right A-comodules.

Now we determine the associated algebraic quantum group (D, Δ_D). Note that we could determine the structure with the help of the cocycle, but we wish to use directly the Galois object itself, since this is easier. In particular, we exploit the pairing between (D, Δ_D) and its dual $(\widehat{D}, \Delta_{\widehat{D}})$.

We first give a heuristic reasoning. We determine the algebra structure of $\widehat{D}$. We need a description of the dual $\widehat{A}$ of $A_\lambda^{n,m}$. It has a basis consisting of expressions $e_p d^q$ with $p \in \mathbb{Z}$ and $0 \leqslant q < n$, where $e_p \in \widehat{A}$ and $d \in M(\widehat{A})$, such that $e_p e_q = \delta_{p,q} e_p$, $d e_p = e_{p-m} d$ and $d^n = 0$. With $c = \sum_k \lambda^{-k} e_k \in M(\widehat{A})$, the comultiplication is determined by

$$\Delta_{\widehat{A}}(e_p) = \sum_t e_t \otimes e_{p-t},$$

$$\Delta_{\widehat{A}}(d) = d \otimes c + 1 \otimes d.$$

Now the left action of $\widehat{A}$ on B is given by

$$\begin{array}{lcll} e_s \cdot x^p y^q & = & \delta_{p,s-mq}\, x^p y^q, & \\ d \cdot x^p y^q & = & C_q\, x^p y^{q-1}, & 0 < q < n, \\ d \cdot x^p & = & 0, & \end{array}$$

where $C_q = \frac{(1-\lambda^{mq})}{(1-\lambda^m)} \lambda^{m(q-1)}$. Consider the operators g_s and h acting on the right of B by

$$\begin{array}{rcll} x^p y^q \cdot g_s & = & \delta_{p,-s}\, x^p y^q, & \\ x^p y^q \cdot h & = & C_q\, x^{p+m} y^{q-1}, & 0 < q < n-1, \\ x^p \cdot h & = & 0. & \end{array}$$

Then it is easy to see that h and g_s commute with the left action of $\widehat{A}$. We see that $h \cdot g_s = g_{s+m} \cdot h$, that $g_s g_t = \delta_{s,t} g_s$ and that $h^n = 0$. The span of $g_s h^q$ will form our algebra $\widehat{D}$. Now denote by $u_{p,q}$ the elements in D such that $\langle u_{p,q}, e_r d^s \rangle = \delta_{p,r}\delta_{q,s}$, and denote $u = u_{-1,0}$, $v = u_{0,0}$ and $w = u_{0,1}$. Then we have $\gamma_B(x) = u \otimes x$ and $\gamma_B(y) = v \otimes y + w \otimes x^m$ by using the action of $\widehat{D}$. Since this has to commute with α_B, we find that $v = 1$. Using that $y^n = \mu x^{mn}$ we find that $\mu + w^n = \mu u^{mn}$, and using $xy = \lambda yx$, we get $uw = \lambda wu$. Furthermore, the fact that x is invertible gives that u is invertible. This then completely determines the structure of D. The coalgebra structure is determined by the usual

$$\Delta_C(u) = u \otimes u,$$

$$\Delta_C(w) = w \otimes u^m + 1 \otimes w.$$

We can now make things exact.

Proposition 4.5.4. *Let D be the unital algebra generated by three elements u, u^{-1} and w, with defining relations: u^{-1} is the inverse of u, $uw = \lambda wu$ and $\mu \cdot 1 + w^n = \mu u^{mn}$. Then D is not trivial. We can define a unital multiplicative comultiplication Δ_D on D, given on the generators by*

$$\Delta_D(u) = u \otimes u,$$

$$\Delta_D(w) = w \otimes u^m + 1 \otimes w,$$

making it an algebraic quantum group of compact type. It has a left coaction γ_B on B determined by

$$\gamma_B(x) = u \otimes x,$$

$$\gamma_B(y) = 1 \otimes y + w \otimes x^m,$$

making it a A-D-bi-Galois object.

Proof. It is easy to see that Δ_D and γ_B can be extended, that Δ_D is coassociative and γ_B a coaction, and that γ_B commutes with the right coaction of A. Since now D is already a bialgebra, it follows from the general theory of Hopf-Galois extensions ([71]) that if γ_B can be shown to make B a left D-Galois object, then automatically D will be a Hopf algebra, hence the reflected algebraic quantum group of A.

We can again show this by explicitly constructing a homomorphism $\beta_D : D \to B \otimes B^{\mathrm{op}}$, where we then also write $\tilde{\beta}_D = (\iota \otimes S_{B^{\mathrm{op}}})\beta_D$ and $\tilde{\beta}_D(c) = c^{[-2]} \otimes c^{[-1]}$. On generators it is given by $\tilde{\beta}_D(u) = x \otimes x^{-1}$ and $\tilde{\beta}_D(w) = y \otimes x^{-m} - 1 \otimes yx^{-m}$. Again the same chore shows that it has a well-defined extension to D, and that it provides the good inverse for the Galois map associated with γ_B. This concludes the proof. □

Remarks: 1. If the characteristic of k is zero, then D will not be isomorphic to any $A_{\lambda'}^{n',m'}$ when $\mu \neq 0$. For in $A_{\lambda'}^{n',m'}$, the only group-like elements are powers of a. Thus any isomorphism would send u to a power a^l of a. But then $\mu(a^{lmn} - 1)$ would have to be an n-th power in A, hence, dividing out by b, also in $k[a, a^{-1}]$. This is impossible.

2. As we have remarked, this example is a cocycle (double) twist construction by a cocycle ω. We have already given the 2-cocycle as a function on $A \otimes A$. But it is also natural to see it as a multiplier of $\widehat{A} \otimes \widehat{A}$. Then we have the expression

$$\omega = 1 \otimes 1 + \mu \sum_{q=0}^{n-1} \frac{1}{(\lambda^m; \lambda^m)_{q-1} \cdot (\lambda^m; \lambda^m)_{n-q-1}} d^q \otimes d^{n-q} c^q,$$

with the notation for the dual as before, and where $(a; z)_k$ denotes the z-shifted factorial ([52]). Now consider the algebra generated by c and d as the fiber at λ^m of the field of algebras on k_0 with the fiber in z generated by c_z, d_z with c_z invertible, $d_z^n = 0$ and $c_z d_z = z d_z c_z$, and with the extra relation $c_z^k = 1$ if z is a primitive k-th root of unity. Then we can formally write

$$\omega = 1 \otimes 1 + \mu \cdot \lim_{z \to \lambda^m} \frac{1}{(z; z)_{n-1}} (d_z \otimes c_z + 1 \otimes d_z)^n,$$

where we take a limit over points which are *not* roots of unity. In this way, since c, d generate a finite-dimensional 2-generator Taft algebra inside $M(\widehat{A})$, we find back a part of the cocycles of [62]. In fact, *any* of those cocycles should give a cocycle inside $M(\widehat{A} \otimes \widehat{A})$, hence a cocycle functional on $A \otimes A$. We have however not carried out the computations in this general case. We remark however that this is a well-known technique to construct cocycle twists (and thus (doubly) cocycle twisted quantum groups), namely considering a 2-cocycle on a substructure, and then lifting this (in a trivial way) to the whole object (see e.g. [33]).

3. There does not seem to be any straightforward modification of the two-generator Taft algebra Galois objects that provides a Galois object for the *dual* of some $A_\lambda^{n,m}$. It would be interesting to see if such non-trivial Galois objects exist.

Analysis

Chapter 5

Preliminaries on von Neumann algebras

In this chapter, we recall some basic notions concerning von Neumann algebras and their weight theory. Most of this material can be found in the standard reference works [83] and [84] (see also [80]).

5.1 von Neumann algebras

We will call a unital *-algebra N a *W*-algebra* or *von Neumann algebra* if there exists a Hilbert space $\mathscr{H}$ and a faithful unital *-homomorphism $\pi : N \to B(\mathscr{H})$, such that the image is closed in the σ-weak topology. By the von Neumann bicommutant theorem, this last condition is equivalent with asking that $\pi(N)$ equals its *bicommutant*: $\pi(N)'' = \pi(N)$, where for a subset $S \subseteq B(\mathscr{H})$, we denote

$$S' = \{x \in B(\mathscr{H}) \mid xs = sx \text{ for all } s \in S\}.$$

We thus neglect the common distinction by which a von Neumann algebra should be seen as a 'concrete W*-algebra' (i.e., a W*-algebra with some fixed faithful *-representation as above), and conversely W*-algebras as 'abstract von Neumann algebras'. For the moment, we will always assume that a von Neumann algebra is represented on *some* fixed Hilbert space $\mathscr{H}$, and we will drop the notation π.

We write the cone of positive elements in a von Neumann algebra N as N^+. We identify its predual N_* with the space of normal (= σ-weakly continuous) functionals on it. It is canonically determined by the fact that

$(N_*)^* = N$ as Banach spaces, and then the σ-weak topology on N is precisely the weak*-topology of N as a dual space of N_*. We denote the positive cone of the predual by N_*^+. When $N_1 \subseteq B(\mathscr{H}_1)$ and $N_2 \subseteq B(\mathscr{H}_2)$ are two (concretely represented) von Neumann algebras, we will denote their spatial tensor product $(N_1 \odot N_2)''$ as $N_1 \otimes N_2 \subseteq B(\mathscr{H}_1 \otimes \mathscr{H}_2)$. Then by Tomita's commutation theorem, we have $(N_1 \otimes N_2)' = N_1' \otimes N_2'$.

We will regularly need to slice with maps: if N_1, N_2 and N_3 are von Neumann algebras, and $\Phi : N_2 \to N_3$ a normal completely positive map, then

$$\iota \otimes \Phi : N_1 \odot N_2 \to N_1 \odot N_3$$

extends uniquely to a normal completely positive map $N_1 \otimes N_2 \to N_1 \otimes N_3$, which we still denote by $\iota \otimes \Phi$.

5.2 Weights on von Neumann algebras

Definition 5.2.1. *Let N be a von Neumann algebra. A* weight *φ on N is a semi-linear*[1] *map*

$$\varphi : N^+ \to [0, +\infty].$$

It is called

1. semi-finite, *if the left ideal $\mathscr{N}_\varphi := \{x \in N \mid \varphi(x^*x) < \infty\}$ of* square integrable elements *is σ-weakly dense in N,*
2. faithful, *if $\varphi(x^*x) = 0$ implies x=0,*
3. normal, *if $\varphi(x) = \lim_i \varphi(x_i)$ for any increasing bounded net $x_i \in N^+$ with $x = \sup x_i$.*

We abbreviate the terminology 'normal semi-finite faithful weight' to 'nsf weight'.

[1]By a semi-linear map $\varphi : N^+ \to [0, +\infty]$, we mean a map such that $\varphi(x + y) = \varphi(x) + \varphi(y)$ for all $x, y \in N^+$, such that $\varphi(rx) = r\varphi(x)$ for $r \in \mathbb{R}_0^+$ and $0 \neq x \in N^+$, and such that $\varphi(0) = 0$.

We introduce some further notation. If N is a von Neumann algebra, and φ a weight on N, we denote by

$$\mathscr{M}_\varphi^+ = \{x \in N^+ \mid \varphi(x) < \infty\}$$

the space of *positive integrable elements*, and by

$$\mathscr{M}_\varphi = \mathscr{N}_\varphi^* \cdot \mathscr{N}_\varphi$$

the space of integrable elements. Then $\mathscr{M}_\varphi$ is the absolutely convex hull of $\mathscr{M}_\varphi^+$, and one can extend φ from $\mathscr{M}_\varphi^+$ to a linear functional $\mathscr{M}_\varphi \to \mathbb{C}$. We will also write φ for this extension.

The following definitions and theorems are very important in the theory of weights on von Neumann algebras (=non-commutative integration theory).

Definition 5.2.2. *Let N be a von Neumann algebra, and φ an nsf weight on N. The Hilbert space completion of $\mathscr{N}_\varphi$ with respect to the inner product*

$$\langle x, y\rangle_\varphi := \varphi(y^* x)$$

is denoted as $\mathscr{L}^2(N,\varphi)$. The inclusion map $\mathscr{N}_\varphi \to \mathscr{L}^2(N,\varphi)$ is denoted as Λ_φ, and is called the GNS map[2] *for φ. There exists a unique unital normal *-representation π_φ of N on $\mathscr{L}^2(N,\varphi)$, called the* GNS representation*, such that*

$$\pi_\varphi(x)\Lambda_\varphi(y) = \Lambda_\varphi(xy)$$

for $x \in N$ and $y \in \mathscr{N}_\varphi$.

The combined triple $(\mathscr{L}^2(N,\varphi), \Lambda_\varphi, \pi_\varphi)$ is called the GNS construction *for (N,φ).*

The GNS construction is a canonical example of a semi-cyclic representation:

Definition 5.2.3. *Let N be a von Neumann algebra. A triple $(\mathscr{H}, \Lambda, \pi)$ is called a* semi-cyclic representation *for N when $\mathscr{H}$ is a Hilbert space, Λ is a linear map $\mathscr{N} \to \mathscr{H}$ with $\mathscr{N} \subseteq N$ a left ideal of N, and π is a normal unital *-representation of N on $\mathscr{H}$, such that Λ has dense range and*

$$\pi(x)\Lambda(y) = \Lambda(xy) \qquad \text{for all } x \in N, y \in \mathscr{N}.$$

[2]GNS is short for Gelfand, Naimark and Segal

When there exists an nsf weight φ *on* N *such that* $\mathscr{N} = \mathscr{N}_\varphi$, *and*

$$\langle \Lambda(x), \Lambda(y) \rangle = \varphi(y^* x) \qquad \text{for all } x, y \in \mathscr{N}_\varphi$$

then we call $(\mathscr{H}, \Lambda, \pi)$ *a* semi-cyclic representation for φ.

When the representation part of a semi-cyclic representation $(\mathscr{H}, \Lambda, \pi)$ is clear from the context, we also call Λ a *semi-cyclic representation (for* φ*) on* $\mathscr{H}$.

The following Definition-Proposition recalls the main parts of the celebrated Tomita-Takesaki theorem.

Definition-Proposition 5.2.4. *Let* N *be a von Neumann algebra,* φ *an nsf weight on* N. *Then the anti-linear map*

$$\mathcal{T}_{\varphi,0} : \Lambda_\varphi(\mathscr{N}_\varphi \cap \mathscr{N}_\varphi^*) \to \Lambda_\varphi(\mathscr{N}_\varphi \cap \mathscr{N}_\varphi^*) : \Lambda_\varphi(x) \to \Lambda_\varphi(x^*)$$

is closable to a (possibly unbounded) anti-linear map $\mathcal{T}_\varphi$, *which is then involutive (i.e. domain and range of* $\mathcal{T}_\varphi$ *are equal, and* $\mathcal{T}_\varphi^2$ *equals the identity on its domain).*

Let $\mathcal{T}_\varphi = J_\varphi \nabla_\varphi^{1/2}$ *be the polar decomposition of* $\mathcal{T}_\varphi$. *Then the positive operator* ∇_φ *is called the* modular operator for φ, *while the anti-unitary* J_φ *is called the* modular conjugation for φ. *They satisfy the commutation relation*

$$J_\varphi \nabla_\varphi^{it} J_\varphi = \nabla_\varphi^{it}.$$

The modular operator induces an $\mathbb{R}$*-parametrized family* σ_t^φ *of* **-automorphisms on* N, *called the* modular automorphism group of φ, *by the formula*

$$\nabla_\varphi^{it} \pi_\varphi(x) \nabla_\varphi^{-it} = \pi_\varphi(\sigma_t^\varphi(x)).$$

Then for $x \in \mathscr{N}_\varphi$, *we have* $\sigma_t^\varphi(x) \in \mathscr{N}_\varphi$ *for any* $t \in \mathbb{R}$, *with*

$$\Lambda_\varphi(\sigma_t^\varphi(x)) = \nabla_\varphi^{it} \Lambda_\varphi(x).$$

The modular conjugation induces a canonical **-isomorphism of* N^{op} *with* $\pi_\varphi(N)'$, *given as*

$$x^{op} \to J_\varphi \pi_\varphi(x)^* J_\varphi.$$

Definition 5.2.5. *Let N be a von Neumann algebra, and φ an nsf weight on N. The* opposite weight of φ *is the nsf weight φ^{op} on the opposite von Neumann algebra N^{op}, given by*

$$\varphi^{op}(x^{op}) = \varphi(x) \qquad \text{for } x \in N^+.$$

We next recall Connes' cocycle derivative theorem from the fundamental paper [16], but we make some preliminary definitions.

Definition 5.2.6. *Let N be a von Neumann algebra. A* one-parametergroup of automorphisms on N *is an $\mathbb{R}$-parametrized set $\{\sigma_t\}$ of *-automorphisms of N, such that $\sigma_{s+t} = \sigma_s \circ \sigma_t$, and such that $t \to \sigma_t$ is point-σ-weakly continuous.*

For example, when φ is an nsf weight on N, the associated modular automorphism group σ_t^φ is a one-parametergroup of automorphisms on N.

Definition 5.2.7. *Let N be a von Neumann algebra, and σ_t a one-parametergroup of automorphisms on N. A* 1-cocycle *for σ_t is an $\mathbb{R}$-parametrized set $\{u_t\}$ of unitaries in N, such that $u_{s+t} = u_s \cdot \sigma_s(u_t)$, and such that $s \to u_s$ is σ-weakly continuous.*

When σ_t and τ_t are two one-parametergroups of automorphisms on N, we call τ_t cocycle equivalent *(or* outer equivalent*) with σ_t when there exists a 1-cocycle u_t for σ_t such that $\tau_t(x) = u_t \cdot \sigma_t(x) \cdot u_t^*$ for all $t \in \mathbb{R}$.*

Theorem 5.2.8. *Let N be a von Neumann algebra, and φ an nsf weight on N.*

If ψ is another nsf weight on N, then σ_t^ψ is cocycle equivalent *with σ_t^φ by a canonically determined 1-cocycle u_t, which is denoted as $(\mathscr{D}\psi : \mathscr{D}\varphi)_t$, and which is called the* cocycle derivative *of ψ with respect to φ.*

Conversely, to any 1-cocycle u_t for σ_t^φ there is uniquely associated an nsf weight ψ such that $u_t = (\mathscr{D}\psi : \mathscr{D}\varphi)_t$.

The following proposition shows, among other things, that a modular conjugation is really associated with N instead of with an associated nsf weight φ. To be able to formulate the proposition rigourously, we introduce some further terminology, whcih will however not be used later on. For φ an nsf weight on a von Neumann algebra N, we denote by $\mathscr{L}^2(N,\varphi)_+$ the *positive cone* of $\mathscr{L}^2(N,\varphi)$, which is the closure of the set of elements of the form $\Delta_\varphi^{1/4}\Lambda_\varphi(x^*x)$ with $x \in \mathscr{N}_\varphi \cap \mathscr{N}_\varphi^*$ (one can show that this expression then makes sense).

Proposition 5.2.9. *Let N be a von Neumann algebra, and φ and ψ two nsf weights on N. Then there exists a unique unitary*

$$U_{\varphi,\psi} : \mathscr{L}^2(N,\varphi) \to \mathscr{L}^2(N,\psi),$$

such that $U_{\varphi,\psi}\pi_\varphi(x)U_{\varphi,\psi}^* = \pi_\psi(x)$ for all $x \in N$ and $U_{\varphi,\psi}\mathscr{L}^2(N,\varphi)_+ = \mathscr{L}^2(N,\psi)_+$. Moreover, we then have that $U_{\varphi,\psi}J_\varphi = J_\psi U_{\varphi,\psi}$.

By the previous proposition, we can in fact canonically identify all GNS spaces of a von Neumann algebra N with a single Hilbert space $\mathscr{L}^2(N)$, in such a way that all GNS representations get transformed into a same representation π_N, and all modular conjugations get transformed into the same anti-unitary J_N. It is then convenient to transport all structure of some $\mathscr{L}^2(N,\varphi)$ to $\mathscr{L}^2(N)$. In particular, we transport Λ_φ to a map $\mathscr{N}_\varphi \to \mathscr{L}^2(N)$, which, by abuse of notation and terminology, we will still denote by Λ_φ, and call the GNS map for φ. We call the triple $(\mathscr{L}^2(N),\Lambda_\varphi,\pi_N)$ the *standard GNS construction* for (N,φ). *We will suppress the notation π_N whenever N is not already identified with some concrete set of operators.* We call the normal unital anti-*-representation

$$\theta_N : N \to B(\mathscr{L}^2(N)) : x \to J_N x^* J_N$$

the *right GNS representation of N*. We also denote

$$C_N : N \to N' \subseteq B(\mathscr{L}^2(N)) : x \to J_N x^* J_N$$

the canonical anti-*-automorphism from N to N'. If φ is an nsf weight on N, we write φ' for the nsf weight $\varphi \circ C_N^{-1}$ on N'.

We will further identify $\mathscr{L}^2(N^{\mathrm{op}})$ with $\mathscr{L}^2(N)$, by choosing an nsf weight φ on N and defining a unitary U which sends $\Lambda_{\varphi^{\mathrm{op}}}(x^{\mathrm{op}})$ with $x \in \mathscr{N}_\varphi^*$ to $J_\varphi\Lambda_\varphi(x^*)$. (One can show that this is independent of the chosen nsf weight φ.) Since N^{op} and N' can be canonically identified by the map $x^{\mathrm{op}} \to J_N x^* J_N$, and since under this isomorphism φ^{op} corresponds to φ', we can and will also identify $\mathscr{L}^2(N')$ with $\mathscr{L}^2(N)$, sending $\Lambda_{\varphi'}(C_N(x))$ to $J_\varphi\Lambda_\varphi(x^*)$ for $x \in \mathscr{N}_\varphi^*$. Finally, we write $\Lambda_\varphi^{\mathrm{op}}$ for the map $\Lambda_{\varphi'} \circ C_N$.

5.3 Analytic extensions of one-parametergroups

Definition 5.3.1. *Let N be a von Neumann algebra, and σ_t a one-parameter-group of automorphisms on N. An element $x \in N$ is called* analytic with respect to σ_t *when for all $\omega \in N_*$, the function $t \to \omega(\sigma_t(x))$ is analytic.*

If $x \in N$ is analytic for σ_t, there exists for each $z \in \mathbb{C}$ a unique element $\sigma_z(x) \in N$ such that, for all $\omega \in N_*$, the complex number $\omega(\sigma_z(x))$ is the value of the analytic extension of $t \to \omega(\sigma_t(x))$ at the point z. The function $z \to \sigma_z(x)$ is called the analytic extension of $t \to \sigma_t(x)$.[3]

For each one-parametergroup of automorphisms, the set of associated analytic elements is always σ-weakly dense in N. In fact, one can easily construct analytic elements by what is called *smoothing*. For example, if $x \in N$, define

$$x_n = \sqrt{\frac{n}{\pi}} \int_{-\infty}^{+\infty} e^{-nt^2} \sigma_t(x) dt,$$

which can be seen as the unique element for which

$$\omega(x_n) = \sqrt{\frac{n}{\pi}} \int_{-\infty}^{+\infty} e^{-nt^2} \omega(\sigma_t(x)) dt$$

for all $\omega \in N_*$. Then x_n is analytic for σ_t, with

$$\sigma_z(x_n) = \sqrt{\frac{n}{\pi}} \int_{-\infty}^{+\infty} e^{-n(t-z)^2} \sigma_t(x) dt,$$

and moreover $x_n \to x$ in the σ-weak topology. (It would be more elegant to use arbitrary kernels in $\mathscr{L}^1(\mathbb{R})$ whose Fourier transform goes to zero quickly enough at infinity, but these concrete forms are sufficient.)

One can show that if $x, y \in N$ are analytic for σ_t, then also xy and x^* are analytic for σ_t, with

$$\sigma_z(xy) = \sigma_z(x)\sigma_z(y),$$

$$\sigma_z(x)^* = \sigma_{\overline{z}}(x^*).$$

We can thus speak about the *-algebra of analytic elements.

[3] Each map σ_z, as defined on analytic elements, is in fact closable (in the (σ-weak)-(σ-weak)-topology), and we then denote the closure by the same symbol. This extension will however rarely be used.

Definition 5.3.2. *Let N be a von Neumann algebra, and φ an nsf weight on N. We call* Tomita *-algebra of φ *(inside N) the *-algebra $\mathscr{T}_\varphi$ of analytic elements x for σ_t^φ for which $\sigma_z^\varphi(x)$ is square integrable for each $z \in \mathbb{C}$.*

The Tomita *-algebra of an nsf weight φ on a von Neumann algebra N is still σ-weakly dense: for example, applying the smoothing process to a square integrable element produces elements in the Tomita *-algebra. We also note that the Tomita *-algebra of an nsf weight really is a sub-*-algebra of N. We further mention that we will also view it as a subspace of $\mathscr{L}^2(N)$ by applying Λ_φ to it (and then call it the Tomita *-algebra *inside* $\mathscr{L}^2(N)$). This last viewpoint is really the original one, as there is also a stand-alone definition of *a* Tomita *-algebra. We will only give the definition *with respect to an nsf weight* (in which case we are again free to work either in the algebra itself or in the associated Hilbert space), and refer to [84], Definition 2.1 for the general definition (which we will only need at one point).

Definition 5.3.3. *Let N be a von Neumann algebra, and φ an nsf weight on N. A* Tomita *-algebra $\mathfrak{A}$ for φ *is a sub-*-algebra of $\mathscr{T}_\varphi$, invariant under all σ_z^φ, such that $\mathfrak{A}$ is σ-weakly dense in N.*

One can show then that $\mathfrak{A}$ is a σ-strong-norm core[4] for Λ_φ (see Theorem VI.1.26 and Proposition VIII.3.15 of [84]).

We now state the most important property of the modular one-parametergroup with regard to the non-tracial character of an arbitrary nsf weight.

Proposition 5.3.4. *(KMS property) Let φ be an nsf weight on a von Neumann algebra N. If $z \in \mathscr{N}_\varphi \cap \mathscr{N}_\varphi^*$ and $y \in \mathscr{T}_\varphi$, then*

$$\varphi(z\sigma_{-i}^\varphi(y)) = \varphi(yz).$$

Proposition 5.3.5. *Let φ be an nsf weight on a von Neumann algebra N. If $y \in \mathscr{T}_\varphi$, then $\sigma_z^\varphi(y) \in \mathscr{N}_\varphi \cap \mathscr{N}_\varphi^*$ for all $z \in \mathbb{C}$, and*

$$J_N\Lambda_\varphi(y) = \Lambda_\varphi(\sigma_{i/2}^\varphi(y)^*).$$

When $z \in \mathscr{N}_\varphi$ and y is analytic for σ_t^φ, then $zy \in \mathscr{N}_\varphi$, and

$$\Lambda_\varphi(zy) = J_N\sigma_{i/2}^\varphi(y)^* J_N\Lambda_\varphi(z).$$

[4] we recall that a core for a closed map between topological vector spaces is a subspace of the domain of the map, such that the graph of the restriction of the map to this subspace is dense in the graph of the map.

We will also need the following proposition (see Theorem VII.2.5 of [84]).

Proposition 5.3.6. *Let N be a von Neumann algebra, and φ an nsf weight on N. Then if $x \in \mathscr{N}_\varphi$ and $z \in \mathscr{N}_{\varphi'}$, we have*

$$z\Lambda_\varphi(x) = x\Lambda_{\varphi'}(z).$$

Conversely, if $z \in N'$, $\xi \in \mathscr{L}^2(N)$ and

$$z\Lambda_\varphi(x) = x\xi$$

for all $x \in \mathscr{N}_\varphi$, then $z \in \mathscr{N}_{\varphi'}$ and $\xi = \Lambda_{\varphi'}(z)$.

5.4 The Connes-Sauvageot tensor product

Most of the discussion in the following two sections is taken from Section IX.3 of [84].

Let π be a unital normal *-representation of a von Neumann algebra N on a Hilbert space $\mathscr{H}$. Let φ be an nsf weight on N. A vector $\xi \in \mathscr{H}$ is called *right bounded* w.r.t. φ and π if the map

$$\Lambda_\varphi(\mathscr{N}_\varphi) \to \mathscr{H} : \Lambda_\varphi(x) \to \pi(x)\xi$$

is bounded, in which case we denote its closure by $R^{\pi,\varphi}(\xi)$ (or R_ξ if π and φ are fixed). We denote by ${}_\varphi\mathscr{H}$ the space of right bounded vectors. Similarly, if θ is a unital normal right *-representation of N, a vector $\xi \in \mathscr{H}$ is called *left bounded* w.r.t. φ if the map

$$\Lambda_\varphi^{\mathrm{op}}(\mathscr{N}_\varphi^*) \to \mathscr{H} : J_N\Lambda_\varphi(x^*) \to \theta(x)\xi$$

is bounded, in which case we denote its closure by $L^{\theta,\varphi}(\xi)$ (or L_ξ if θ and φ are fixed). We denote by $\mathscr{H}_\varphi$ the space of left bounded vectors for θ. Remark that if $\pi' = \theta \circ C_N$ is the associated left *-representation of N', then the right bounded vectors with respect to φ' are exactly the left bounded vectors with respect to φ.

Now let N be a von Neumann algebra, and φ an arbitrary nsf weight on N. If θ is a unital normal right *-representation of N on a Hilbert space $\mathscr{G}$, and π a unital normal *-representation of N on a Hilbert space $\mathscr{H}$, we denote by

$\mathscr{G}\,{}_{\theta}\underset{\varphi}{\otimes}{}_{\pi}\,\mathscr{H}$ (or simply $\mathscr{G}\underset{\varphi}{\otimes}\mathscr{H}$ when θ and π are clear) their Connes-Sauvageot tensor product with respect to π, θ and φ. It is the Hilbert space closure of the algebraic tensor product of $\mathscr{G}_{\varphi}$ and $\mathscr{H}$ with respect to the scalar product

$$\langle \xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2 \rangle_{\mathrm{CS}} \quad := \quad \langle \pi(L_{\eta_1}^* L_{\xi_1})\xi_2, \eta_2 \rangle,$$

modulo vectors of norm zero. In fact, we could as well start with the algebraic tensor product of $\mathscr{G}_{\varphi}$ and ${}_{\varphi}\mathscr{H}$, since the image of this tensor product in the previous Hilbert space will be dense. On elementary tensors of the last space, we can give a different form of the scalar product, namely

$$\langle \xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2 \rangle \quad = \quad \langle \theta(R_{\eta_2}^* R_{\xi_2})\xi_1, \eta_1 \rangle.$$

The image of an elementary tensor in $\mathscr{G} \underset{\varphi}{\otimes} \mathscr{H}$ will then be denoted by the same symbol, with $\otimes$ replaced by ${}_{\theta}\underset{\varphi}{\otimes}{}_{\pi}$ or simply $\underset{\varphi}{\otimes}$.[5]

If then $x \in \theta(N)'$ and $y \in \pi(N)'$, and $\eta \in {}_{\varphi}\mathscr{H}$, also $y\eta \in {}_{\varphi}\mathscr{H}$, and one can form an operator $x \underset{\varphi}{\otimes} y$ on $\mathscr{G} \underset{\varphi}{\otimes} \mathscr{H}$, uniquely determined by the fact that

$$(x \underset{\varphi}{\otimes} y)(\xi \underset{\varphi}{\otimes} \eta) = (x\xi) \underset{\varphi}{\otimes} (y\eta)$$

for $\xi \in \mathscr{G}$ and $\eta \in {}_{\varphi}\mathscr{H}$.

In the beginning of the eleventh chapter, we will need the notion of a fibre product of two von Neumann algebras over a third von Neumann algebra. This is a von Neumann algebraic version of an algebraic construction already commented upon at the end of the first chapter. Namely, suppose L, X and Y are three unital algebras, s_L a unital anti-homomorphism from L to X, and t_L a unital homomorphism from L to Y. Then X can be seen as a right (resp. left) L-module by considering the left (resp. right) L^{op}-module structure on X induced by left (resp. right) multiplication composed with s_L, while Y can be seen as a left (resp. right) L-module by left (resp. right) multiplication composed with t_L. Then $X \underset{L}{\odot} Y$ is still an L-L-bimodule, and we can consider the central elements $(X \underset{L}{\odot} Y)^L$. These will then form an algebra under factor-wise multiplication. It is this construction which is

[5] In fact, there is also a *weight-independent* definition of the Connes-Sauvageot tensor product, and all weight-dependent constructions can be canonically identified with it, preserving all further structure.

'generalized' to the von Neumann algebra setting.

So let L, N_1 and N_2 be von Neumann algebras, s a normal unital anti-*-homomorphism from L to N_1, and t a normal unital *-homomorphism from L to N_2. Let π_i be a unital normal left *-representation of N_i on a Hilbert space $\mathscr{H}_i$. Then by restricting to L via s and t, we also obtain a normal right L-representation θ on $\mathscr{H}_1$ and a normal left L-representation π on $\mathscr{H}_2$. Let μ be an arbitrary nsf weight on L, and let $\mathscr{H}_1 \underset{\mu}{\otimes} \mathscr{H}_2$ be the Connes-Sauvageot tensor product. Then since $\theta(L) \subseteq \pi_1(N_1)$ and $\pi(L) \subseteq \pi_2(N_2)$, we can represent the *commutants* of $\pi_1(N_1)$ and $\pi_2(N_2)$ on $\mathscr{H}_1 \underset{\mu}{\otimes} \mathscr{H}_2$. We then define the von Neumann algebra $N_1 \underset{L}{{}_s{*}_t} N_2$ as the von Neumann algebra consisting of operators on $\mathscr{H}_1 \underset{\mu}{\otimes} \mathscr{H}_2$ which commute elementwise with the images of these commutants. One can show that $N_1 \underset{L}{{}_s{*}_t} N_2$, as a von Neumann algebra, is in fact independent of the choices made along the way, and only depends on s and t. It is called the *fibre product* of N_1 and N_2 over L.

One can then perform on these fibre products most slice constructions as for ordinary tensor products (which is the case $L = \mathbb{C}$). For example, one can slice with functionals and nsf weights, one can slice with *-homomorphisms if they are well-behaved with respect to the base algebra L, one can slice with operator valued weights if they are well-behaved with respect to the base algebra, etc. Such a slice is then denoted for example as $\iota \underset{L}{{}_s{*}_t} -$. We refer to the introduction of [30] for some more concrete information. We will in fact only need this construction in a very special case, for which these slice constructions greatly simplify.

5.5 Morita theory for von Neumann algebras and weights

Definition 5.5.1. *Let M and P be von Neumann algebras. A P-M-*correspondence *$(\mathscr{H}, \pi, \theta)$ is a triple consisting of a Hilbert space $\mathscr{H}$ with a normal unital *-representation π of P and a normal unital anti-*-representation θ of M, such that $\pi(P) \subseteq \theta(M)'$. We call $(\mathscr{H}, \pi, \theta)$ a P-M-*equivalence correspondence *when π and θ are faithful, and $\theta(M)' = \pi(P)$.*

When the maps π and θ are clear from the context, we denote the correspondence just by $\mathscr{H}$.

On the algebraic side, these correspond to the equivalence bimodules of Definition 1.1.11. The following structure then corresponds to the linking algebras of Definition 1.1.9.

Definition 5.5.2. *A* linking von Neumann algebra *consists of a couple (Q, e), where Q is a von Neumann algebra and e is a (self-adjoint) projection in Q, such that both e and $(1_Q - e)$ are full*[6] *(i.e. the two-sided ideals generated by e and $(1_Q - e)$ both have Q as their σ-weak closure, i.e. their central support is 1_Q).*

If M and P are two von Neumann algebras, we call a quadruple (Q, e, Φ_M, Φ_P) a linking von Neumann algebra between M and P *if (Q, e) is a linking algebra and Φ_M is a *-isomorphism from M to eQe, and Φ_P from P to $(1_Q - e)Q(1_Q - e)$.*

If M is a von Neumann algebra, θ_1 and θ_2 two unital normal right anti-*-representations of M on Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, we call $(\theta_1 \oplus \theta_2)(M)' \subseteq B(\begin{pmatrix} \mathscr{H}_1 \\ \mathscr{H}_2 \end{pmatrix})$, together with the projection onto $\mathscr{H}_2$, the linking von Neumann algebra between the right representations θ_1 and θ_2.

It can be shown that a linking von Neumann algebra between right representations really is a linking von Neumann algebra.

We will further write a linking von Neumann algebra (Q, e) as $\begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}$ as well as $\begin{pmatrix} P & N \\ O & M \end{pmatrix}$, and we identify each Q_{ij} with its part in Q. We also keep the same conventions as in the algebraic setting (for example, the one following Definition 1.1.9).

As in the purely algebraic case, there is a one-to-one correspondence between (isomorphism classes of) equivalence correspondences and (isomorphism classes of) linking von Neumann algebras between.

Definition 5.5.3. *Let M and P be von Neumann algebras, and $(\mathscr{H}, \pi, \theta)$ an P-M-equivalence bimodule. Then we call the linking von Neumann algebra*

[6] Note that the fullness here differs from the purely algebraical definition, since we use the σ-weak topology on Q.

between θ *and the standard right GNS-representation for* M *the* linking von Neumann algebra (between) associated to $\mathscr{H}$. *We then denote its canonical representation on* $\begin{pmatrix} \mathscr{H} \\ \mathscr{L}^2(M) \end{pmatrix}$ *by* π^2 *(or* π^2_Q*).*

The fact that the above Q is a linking von Neumann algebra (between) of course requires proof, but the main ingredients are provided (for example) in section IX.3 of [84]. Also, it is better to see Q as an abstract von Neumann algebra, and π^2 as a concrete representation, for reasons which will soon become clear.

To produce an equivalence correspondence from a linking von Neumann algebra (between), we need to introduce some further terminology.

Definition 5.5.4. *Let* (Q, e) *be a linking von Neumann algebra between von Neumann algebras* M *and* P. *If* φ_P *is a weight on* P *and* φ_M *a weight on* M, *the* balanced weight $\varphi_P \oplus \varphi_M$ *is the weight*

$$\varphi_P \oplus \varphi_M : Q^+ \to [0, +\infty] : \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \to \varphi_P(x_{11}) + \varphi_M(x_{22}).$$

If (Q, e) is a linking von Neumann algebra between von Neumann algebras M and P, and φ_M and φ_P nsf weights on respectively M and P, then their balanced weight will again be nsf. Denote

$$\mathscr{L}^2(Q_{ij}) = \pi_Q(e_i)\theta_Q(e_j)\mathscr{L}^2(Q),$$

where $e_1 = 1_Q - e$ and $e_2 = e$. Then one can write

$$\mathscr{L}^2(Q) = \begin{pmatrix} \mathscr{L}^2(Q_{11}) & \mathscr{L}^2(Q_{12}) \\ \mathscr{L}^2(Q_{21}) & \mathscr{L}^2(Q_{22}) \end{pmatrix},$$

where the expression on the right is just a direct sum of Hilbert spaces, written as a direct sum to indicate how Q acts on it from the left. Now

$$x \in \mathscr{N}_{\varphi_M} \to \mathscr{L}^2(Q_{22}) : x \to \Lambda_{\varphi_P \oplus \varphi_M}(x)$$

will determine a semi-cyclic representation for φ_M, and we can identify $\mathscr{L}^2(M)$ with $\mathscr{L}^2(Q_{22})$ in this way. One can show that this identification is in fact independent of φ_M. In the same way, $\mathscr{L}^2(P)$ can be identified with $\mathscr{L}^2(Q_{11})$. Then $\mathscr{L}^2(Q_{12})$, which we will also denote as $\mathscr{L}^2(N)$ (and $\mathscr{L}^2(Q_{21})$ as $\mathscr{L}^2(O)$), is a P-M-equivalence-correspondence in a natural way, which we call the *P-M-equivalence correspondence associated to* (Q, e).

When Q was in fact the linking von Neumann algebra associated to an equivalence correspondence $\mathscr{H}$, then $\mathscr{L}^2(Q_{12})$ and $\mathscr{H}$ can be canonically identified as equivalence correspondences. For this, one proves that, for some fixed nsf weights φ_M and φ_P on resp. M and P, the space of elements L_ξ, where $\xi \in \mathscr{H}$ ranges over the left bounded vectors for φ_M in $\mathscr{H}$, is precisely $\mathscr{N}_{\varphi_P \oplus \varphi_M} \cap Q_{12}$, so that an identification is provided by sending $\Lambda_{\varphi_P\oplus\varphi_M}(L_\xi)$ to ξ (which is in fact independent of the choice of weights). Conversely, the linking von Neumann algebra associated to the equivalence correspondence of a linking von Neumann algebra Q will be Q itself, represented on the $\begin{pmatrix} \mathscr{L}^2(Q_{12}) \\ \mathscr{L}^2(Q_{22}) \end{pmatrix}$-part of $\mathscr{L}^2(Q)$.

In the following, we will then always identify an equivalence correspondence $\mathscr{H}$ with its part inside $\mathscr{L}^2(Q)$.

We introduce some further notation concerning the GNS representation for a linking von Neumann algebra. We will denote by π_{ik}^j the (faithful) representation of Q_{ik} as maps $\mathscr{L}^2(Q_{kj}) \to \mathscr{L}^2(Q_{ij})$ for $i, j, k \in \{1, 2\}$. We denote by π_Q^j the representation of Q on $\begin{pmatrix} \mathscr{L}^2(Q_{1j}) \\ \mathscr{L}^2(Q_{2j}) \end{pmatrix}$, and by π_Q the standard representation of Q on $\mathscr{L}^2(Q)$. We will use these representation symbols as much as possible to avoid confusion, but we will suppress them when it would muddle up formulas. The GNS map of $\varphi_P \oplus \varphi_M$, restricted to $\mathscr{N}_{\varphi_P\oplus\varphi_M} \cap Q_{ij}$, will be denoted by Λ_{ij} (when φ_P and φ_M are clear from the context). We use the similar notation for the splitting of the standard right representation θ_Q, the i in θ_Q^i now denoting the *row* on which is acted, and θ_{ij}^k being the right representation of Q_{ij} as maps from $\mathscr{L}^2(Q_{ki})$ to $\mathscr{L}^2(Q_{kj})$.

We now comment on the modular structure of an nsf balanced weight on a linking von Neumann algebra. First, we will have that $\nabla_{\varphi_P\oplus\varphi_M}^{it}$ restricts to one-parametergroups of unitaries on each $\mathscr{L}^2(Q_{ij})$, which on the $\mathscr{L}^2(Q_{ii})$-parts coincide with the one-parametergroups of unitaries associated to the modular operator of $\varphi_{Q_{ii}}$. We call the restriction of $\nabla_{\varphi_P\oplus\varphi_M}$ to $\mathscr{L}^2(Q_{12})$ the *spatial derivative of ψ by φ'*, and denote it by $\frac{d\psi}{d\varphi'}$.
Second, the modular conjugation J_Q restricts to the $\mathscr{L}^2(Q_{ii})$-parts, coinciding there with the modular conjugations $J_{Q_{ii}}$, while it sends the $\mathscr{L}^2(Q_{12})$-part to the $\mathscr{L}^2(Q_{21})$-part by an anti-unitary J_N, and vice versa, the $\mathscr{L}^2(Q_{21})$-part to the $\mathscr{L}^2(Q_{12})$-part via an anti-unitary J_O, so that $J_O J_N$ and $J_N J_O$

both equal the identity. This allows us to canonically identify $\mathscr{L}^2(Q_{21})$ with $\overline{\mathscr{L}^2(Q_{12})}$, identifying $J_N\xi$ with $\overline{\xi}$ for $\xi \in \mathscr{L}^2(Q_{12})$.

The following proposition characterizes spatial derivatives with respect to a fixed weight.

Proposition 5.5.5. *Let M be a von Neumann algebra, and $\mathscr{H}$ a P-M-equivalence correspondence, denoting the associated right M-representation by θ. Let φ_M be an nsf weight on M. Suppose ∇^{it} is a one-parametergroup of unitaries on $\mathscr{H}$ such that $\nabla^{it}\theta(x)\nabla^{-it} = \theta(\sigma_t^{\varphi_M}(x))$ for all $x \in M$. Then there exists a unique nsf weight φ_P on P such that $(\frac{d\varphi_P}{d\varphi'_M})^{it} = \nabla^{it}$.*

Proof. This follows from Theorem IX.3.11, Proposition IX.3.10.(i) and Proposition IX.3.8.(i) in [84]. □

If $\mathscr{H}$ is a P-M-equivalence correspondence, we can also be more specific about the map Λ_{21} (relative to fixed nsf weights on P and M): it has a core consisting of elements L_ξ^* where $\xi \in \mathscr{H} \cong \mathscr{L}^2(Q_{12})$ is left bounded *and* in the domain of $(\frac{d\varphi_P}{d\varphi'_M})^{1/2}$. On such elements, we then have $\Lambda_{21}(L_\xi^*) = J_{\mathscr{H}}(\frac{d\varphi_P}{d\varphi'_M})^{1/2}\xi$, where $J_{\mathscr{H}}$ denotes the canonical conjugation $\mathscr{H} \to \overline{\mathscr{H}} \cong \mathscr{L}^2(Q_{21})$. In fact, this observation is *used* to construct the nsf weight φ_P as in the above proposition (see Lemma IX.3.12 in [84]).

We introduce some further terminology. When $\mathscr{H}$ is a Hilbert space, denote by $C_{\mathscr{H}}$ the canonical anti-*-isomorphism $B(\mathscr{H}) \to B(\overline{\mathscr{H}})$, which sends x to $J_{\mathscr{H}}x^*J_{\mathscr{H}}^{-1}$.

Definition 5.5.6. *Let M be a von Neumann algebra. Then we call*

$$(\mathscr{L}^2(M), \pi_M, \theta_M)$$

the identity equivalence correspondence.

If $\mathscr{H}$ is a P-M(-equivalence) correspondence $(\mathscr{H}, \pi, \theta)$, we call the M-P(-equivalence) correspondence $(\overline{\mathscr{H}}, C_{\mathscr{H}} \circ \theta, C_{\mathscr{H}} \circ \pi)$ the conjugate *(or also, in the case of equivalence correspondences, the* inverse*) (equivalence) correspondence of $\mathscr{H}$.*

If M_1, M_2 and M_3 are three von Neumann algebras, $(\mathscr{H}_1, \pi_1, \theta_1)$ an M_1-M_2-equivalence correspondence and $(\mathscr{H}_2, \pi_2, \theta_2)$ a M_2-M_3-equivalence correspondence, and φ an nsf weight on M_2, then we call

$$(\mathscr{H}_1 \underset{\varphi}{\otimes} \mathscr{H}_2, \pi_1(\,\cdot\,) \underset{\varphi}{\otimes} 1, 1 \underset{\varphi}{\otimes} \theta_2(\,\cdot\,))$$

the composite *M_3-M_1-equivalence correspondence of $\mathscr{H}_1$ and $\mathscr{H}_2$.*

One can show that the composite equivalence correspondence is a genuine equivalence correspondence between M_3 and M_1.

Inside a linking von Neumann algebra, the identification of $\mathscr{L}^2(Q_{21})$ with $\overline{\mathscr{L}^2(Q_{12})}$ by (a part of) the modular conjugation, is then in fact an identification of $\mathscr{L}^2(Q_{21})$ with the conjugate correspondence of $\mathscr{L}^2(Q_{12})$.

One has corresponding definitions for linking von Neumann algebras.

Definition 5.5.7. *Let M be a von Neumann algebra. Then we call $(M \otimes M_2(\mathbb{C}), 1_M \otimes e_{22})$ the* identity *linking von Neumann algebra (between M and itself).*

If (Q, e) is a linking von Neumann algebra (between), then we call $(Q, 1_Q - e)$ the inverse *linking von Neumann algebra (between).*

If M_1, M_2 and M_3 are three von Neumann algebras, Q_1 a linking von Neumann algebra between M_2 and M_1, and Q_2 a linking von Neumann algebra between M_3 and M_2, we call

$$\tilde{Q} = (\theta^1_{Q_{1,22}} \oplus \theta_{M_2} \oplus \theta^2_{Q_{2,11}})(M_2)' \subseteq B(\begin{pmatrix} \mathscr{L}^2(Q_{1,12}) \\ \mathscr{L}^2(M_2) \\ \mathscr{L}^2(Q_{2,21}) \end{pmatrix})$$

the associated 3×3-linking von Neumann algebra, and denote this particular representation on $\begin{pmatrix} \mathscr{L}^2(Q_{1,12}) \\ \mathscr{L}^2(M_2) \\ \mathscr{L}^2(Q_{2,21}) \end{pmatrix}$ by $\pi^2_{\tilde{Q}}$. We call the von Neumann algebra constituted by the corners of this 3×3-linking von Neumann algebra the composite linking von Neumann algebra (between) *of Q_2 and Q_1.*

We mention that, in the notation of the above definition, we have a canonical isomorphism $\mathscr{L}^2(\tilde{Q}_{13}) \cong \mathscr{L}^2(\tilde{Q}_{12}) \underset{\varphi}{\otimes} \mathscr{L}^2(\tilde{Q}_{23})$, for any nsf weight φ on

M_2, where we of course write $\tilde{Q} = \begin{pmatrix} \tilde{Q}_{11} & \tilde{Q}_{12} & \tilde{Q}_{13} \\ \tilde{Q}_{21} & \tilde{Q}_{22} & \tilde{Q}_{23} \\ \tilde{Q}_{31} & \tilde{Q}_{32} & \tilde{Q}_{33} \end{pmatrix}$. So the correspondence between equivalence correspondences and linking von Neumann algebras preserves composition.

For further reference, we restate part of the preceding definition in a different way.

Lemma 5.5.8. *Let M_1, M_2 and M_3 be von Neumann algebras. Let $\mathscr{H}_1$ be an M_1-M_2-equivalence correspondence, and let $\mathscr{H}_2$ be an M_3-M_2-equivalence correspondence. Then the commutant of the direct sum right representation is the composite of the linking von Neumann algebra between $\mathscr{L}^2(M_2)$ and $\overline{\mathscr{H}}_2$ and the one between $\mathscr{H}_1$ and $\mathscr{L}^2(M_2)$.*

We end with the following definitions.

Definition 5.5.9. *If M and P are two von Neumann algebras, we call them* W*-Morita equivalent*, if there exists a linking von Neumann algebra between them, or equivalently, if there exists a P-M-equivalence correspondence.*

By the operations on equivalence correspondences, introduced in Definition 5.5.6, one sees that this defines an equivalence relation between von Neumann algebras. Further, we have that isomorphism classes of equivalence correspondences again provide morphisms in a certain large groupoid with von Neumann algebras as objects, the identity equivalence correspondences providing units, and the inverse of an equivalence correspondence giving the inverse of a morphism.

We also briefly introduce the corresponding notions in the C*-algebra context.

Definition 5.5.10. *A* linking C*-algebra *is a couple (E, e) consisting of a C*-algebra E and a (self-adjoint) projection $e \in M(E)$ such that e and $1_E - e$ are full (i.e. EeE and $E(1_E - e)E$ are norm-dense in E).*

When A and D are two C*-algebras, a linking C*-algebra between A and D *is a linking C*-algebra (E, e) with fixed *-isomorphisms between A and* *eEe, and D and* $(1_E - e)E(1_E - e)$.

If A and D are two C-algebras between which there exists a linking C*-algebra, we call them* C*-Morita equivalent.

In the literature, C*-Morita equivalence is called *strong Morita equivalence* (cf. [67]), but we will use the above term for conformity.

5.6 Operator valued weights

The following definitions and results are obtained from sections IX.4 of [84] and section 10 of [31].

We recall the definition of the extended positive cone of a von Neumann algebra (Def. IX.4.4 in [84]).

Definition 5.6.1. *Let N be a von Neumann algebra. The* extended positive cone $N^{+,\mathrm{ext}}$ *of N consists of all semi-linear maps $N_*^+ \to [0,+\infty]$ which are lower semi-continuous w.r.t. the norm-topology on N_*.*

If $x \in N^{+,\mathrm{ext}}$, we write the evaluation of x in $\omega \in N_*^+$ as $\omega(x)$. We will always identify N^+ with its part inside $N^{+,\mathrm{ext}}$.

The following is Definition IX.4.12 of [84].

Definition 5.6.2. *Let $N_0 \subseteq N$ be a unital normal inclusion of von Neumann algebras. An* operator-valued weight T from N to N_0 *(or* N_0-valued weight on N*) is a semi-linear map*

$$T : N^+ \to (N_0)^{+,ext}$$

such that

$$T(y^*xy) = y^*T(x)y \qquad \forall x \in N^+, y \in N_0.$$

Operator-valued weights are natural generalizations of weights, which correspond to the case $N_0 = \mathbb{C}$ (identifying $\mathbb{C}$ with $\mathbb{C} \cdot 1_N$). Then if T is an operator-valued weight from N to N_0, we can define sets $\mathscr{N}_T, \mathscr{M}_T^+$ and $\mathscr{M}_T$ in completely the same way as for ordinary weights. Also the notions of a 'semi-finite' or 'faithful' operator valued weight are immediately clear. As for the normalcy of an operator valued weight: we call an operator valued weight T from N to N_0 *normal* if

$$\omega(T(x)) = \lim_i \omega(T(x_i))$$

for all $\omega \in N_*^+$ and $x_i, x \in N^+$ for which x_i is a bounded increasing net with $x = \sup x_i$. A (normal) operator-valued weight for which $T(1_N) = 1_{N_0}$ is called a *(normal) conditional expectation*; it then automatically satisfies $\mathscr{M}_T = N$.

One can always extend an nsf operator valued weight T from N to N_0 uniquely to a semi-linear map $N^{+,\mathrm{ext}} \to N_0^{+,\mathrm{ext}}$. This extension of T, which we denote by the same symbol, will then be a surjective map. This also provides us with a straightforward way of composing nsf operator valued weights: when $N_0 \subseteq N \subseteq N_2$ are unital normal inclusions of von Neumann algebras, T_2 an nsf N-valued weight on N_2 and T an nsf N_0-valued weight on N, we obtain an nsf N_0-valued weight $T \circ T_2$ on N_2 by

$$(T \circ T_2)(x) := T(T_2(x)) \qquad \text{for } x \in N_2^+.$$

In particular, we can compose an nsf operator valued weight T from N_1 to N_0 with an nsf weight on N_0 to obtain an nsf weight on N_1.

Proposition 5.6.3. *Let $N_0 \subseteq N$ be a normal unital inclusion of von Neumann algebras. Let $T : N \to N_0$ be an nsf N_0-valued weight on N. Then for any nsf weight μ on N_0, we have that $\varphi := \mu \circ T$ is an nsf weight on N, whose modular one-parametergroup σ_t^φ restricts to σ_t^μ on N_0:*

$$\sigma_t^\varphi(\pi_N(x)) = \pi_N(\sigma_t^\mu(x)) \qquad \textit{for all } x \in N_0.$$

Moreover, the restriction σ_t^T of σ_t^φ to $N_0' \cap N$ is independent of the choice of weight μ on N_0, and is called the modular one-parametergroup of T *on $N_0' \cap N$.*

Now let $N_0 \subseteq N$ be a normal unital inclusion of von Neumann algebras, and T an nsf N_0-valued weight on N. Let μ be a fixed nsf weight on N_0, and denote by φ the nsf weight $\mu \circ T$. Consider $x \in \mathscr{N}_T$. Then $xy \in \mathscr{N}_\varphi$ when $y \in \mathscr{N}_\mu$, and

$$\Lambda_\mu(y) \to \Lambda_\varphi(xy)$$

extends from $\Lambda_\mu(\mathscr{N}_\mu)$ to a bounded operator $\mathscr{L}^2(N_0) \to \mathscr{L}^2(N)$, which we will denote by

$$\Lambda_T(x) : \mathscr{L}^2(N_0) \to \mathscr{L}^2(N),$$

following the notations of Theorem 10.6 of [31]. Its adjoint is then determined by

$$\Lambda_T(x)^*\Lambda_\varphi(y) = \Lambda_\mu(T(x^*y)), \qquad \text{for } y \in \mathscr{N}_\varphi \cap \mathscr{N}_T.$$

One can show that

$$\Lambda_T : \mathscr{N}_T \to B(\mathscr{L}^2(N_0), \mathscr{L}^2(N))$$

is independent of the choice of μ.

There is a slight ambiguity of notation now, as $\Lambda_\varphi(x)$ denotes either an element of $\mathscr{H}$ or a linear operator $\mathbb{C} \to \mathscr{H}$. This ambiguity is easily resolved by identifying the Hilbert spaces $B(\mathbb{C}, \mathscr{H})$ and $\mathscr{H}$ by sending x to $x \cdot 1$, and we will make this identification when necessary without further comment.

The theory of operator-valued weights provides a framework in which the tensor products of *nsf* weights can be easily treated.

Definition 5.6.4. *Let N_1 and N_2 be von Neumann algebras, φ_1 an nsf weight on N_1 and φ_2 an nsf weight on N_2. Denote by $(\iota \otimes \varphi_2)$ the map $(N_1 \bar{\otimes} N_2)^+ \to (N_1)^{+,ext}$ which sends $x \in (N_1 \bar{\otimes} N_2)^+$ to the element $(\iota \bar{\otimes} \varphi_2)(x)$ such that for $\omega \in (N_1)_*^+$, we have*

$$\omega((\iota \otimes \varphi_2)(x)) = \varphi_2((\omega \otimes \iota)(x)).$$

Then $(\iota \otimes \varphi_2)$ is an nsf operator valued weight from $N_1 \bar{\otimes} N_2$ to $N_1 (\cong N_1 \otimes 1)$).

Similarly, $(\varphi_1 \otimes \iota)$ can be made sense of as an operator valued weight $(N_1 \otimes N_2)^+ \to (N_2)^{+,ext}$, and then $\varphi_2 \circ (\varphi_1 \otimes \iota) = \varphi_1 \circ (\iota \otimes \varphi_2)$. We denote this composition as $\varphi_1 \otimes \varphi_2$, and call it the tensor product weight *of φ_1 and φ_2.*

One can show that in the situation of the previous definition, $\mathscr{N}_{\varphi_1} \odot \mathscr{N}_{\varphi_2}$ is a core for $\Lambda_{\varphi_1 \otimes \varphi_2}$, and that

$$\Lambda_{\varphi_1} \otimes \Lambda_{\varphi_2} : \mathscr{N}_{\varphi_1} \odot \mathscr{N}_{\varphi_2} \to \mathscr{L}^2(N_1) \otimes \mathscr{L}^2(N_2) :$$

$$x \otimes y \to \Lambda_{\varphi_1}(x) \otimes \Lambda_{\varphi_2}(y)$$

extends to a semi-cyclic representation for $\varphi_1 \otimes \varphi_2$. This provides an identification of $\mathscr{L}^2(N_1 \otimes N_2)$ with $\mathscr{L}^2(N_1) \otimes \mathscr{L}^2(N_2)$ as $N_1 \otimes N_2$-equivalence correspondences, which is in fact independent of the choice of weights. In the following, we will always identify these two spaces without further comment. We further note that the modular conjugation then becomes the tensor product of the respective modular conjugations.

We will also need the following lemma at a certain point.

Lemma 5.6.5. *Let M be a von Neumann subalgebra of $B(\mathscr{H})$ for some Hilbert space $\mathscr{H}$, containing the identity operator. If there exists a faithful normal conditional expectation $\mathcal{E} : B(\mathscr{H}) \to M$, then M is atomic, i.e. a von Neumann algebraic direct sum of type I-factors: there exists an orthogonal central partition p_i of the unity of M, such that each $p_i M$ is a type I-factor (i.e. *-isomorphic to $B(\mathscr{H}_i)$ for some Hilbert space $\mathscr{H}_i$).*

Proof. This is a special situation of Exercise IX.4.1 (parts *d*) and *e*)) in [84]. □

5.7 The basic construction

Remark: The discussion here is borrowed from the paper [20]. We are unaware of this material being treated explicitly in the literature, although we strongly believe that these results, which are generalizations of very well-known ones in particular cases (see e.g. [49], Proposition 4.4.1.(ii)) are known to the experts.

Definition 5.7.1. *Let $N_0 \subseteq N$ be a unital normal inclusion of von Neumann algebras. Let θ_N be the right GNS representation of N, and denote $N_2 = \theta_N(N_0)'$. Then it is immediate that*

$$N_0 \subseteq N \subseteq N_2.$$

We call this string of inclusions (or also just the von Neumann algebra N_2) the basic construction for the inclusion $N_0 \subseteq N$. *Iterating this construction, we obtain a sequence*

$$N_0 \subseteq N \subseteq N_2 \subseteq N_3 \subseteq \ldots,$$

called the (Jones) tower associated with $N_0 \subseteq N$.

Let $N_0 \subseteq N$ be a unital normal inclusion of von Neumann algebras. Then $\mathscr{L}^2(N)$ is an N_0-N_0-correspondence by restricting π_N and θ_N to N_0. We note that it is isomorphic to its conjugate correspondence by the map

$J_{\mathscr{L}^2(N)}J_N$. Let further μ be an nsf weight on N_0. If $N_0 \subseteq N \subseteq N_2$ is the basic construction, then by the construction of N_2 and the theory of section 5.5, we have a canonical unitary N_2-N_2-bimodule map from $\mathscr{L}^2(N) \underset{\mu}{\otimes} \overline{\mathscr{L}^2(N)}$ to $\mathscr{L}^2(N_2)$, and hence also from $\mathscr{L}^2(N) \underset{\mu}{\otimes} \mathscr{L}^2(N)$ to $\mathscr{L}^2(N_2)$. We want to show now that in the presence of an N_0-valued nsf weight on N, this unitary can be expressed in terms of N (Theorem 5.7.5).[7]

In the following, we fix a unital normal inclusion of von Neumann algebras $N_0 \subseteq N$, an nsf weight μ on N_0, and an nsf operator valued weight T from N to N_0. We denote by φ the nsf weight $\mu \circ T$ on N, and by $N_0 \subseteq N \subseteq N_2$ the basic construction. The operators of the form $\Lambda_T(x)\Lambda_T(y)^*$, with $x, y \in \mathscr{N}_T$, will generate a σ-weakly dense sub-*-algebra of N_2, and then there exists a unique N-valued nsf weight T_2 on N_2, such that $\Lambda_T(x)\Lambda_T(y)^* \in \mathscr{M}_{T_2}$ for $x, y \in \mathscr{N}_T$, with

$$T_2(\Lambda_T(x)\Lambda_T(y)^*) = xy^*$$

(cf. [31], Theorem 10.7). We call T_2 the *basic construction for* T, and also write this as

$$N_0 \underset{T}{\subseteq} N \underset{T_2}{\subseteq} N_2.$$

Iterating this construction, we obtain a sequence

$$N_0 \underset{T}{\subseteq} N \underset{T_2}{\subseteq} N_2 \underset{T_3}{\subseteq} N_3 \underset{T_4}{\subseteq} \ldots,$$

called the *tower construction for* T. We then also call $\varphi = \mu \circ T$, $\varphi_2 = \varphi \circ T_2$, ... the *tower construction for* μ *w.r.t.* T.

We remark that φ_2 will then equal the unique nsf weight which satisfies $\frac{d\varphi_2}{d\mu'} = \nabla_\varphi$ (cf. section 10 of [31]).

We prove a lemma about interchanging the analytic continuation of a modular one-parametergroup with an operator valued weight.

Lemma 5.7.2. *Let Q be the linking algebra between the normal right N_0-representations on $\mathscr{L}^2(N)$ and $\mathscr{L}^2(N_0)$, and consider the balanced weight $\varphi_Q := \varphi_2 \oplus \mu$ on Q. Let $x \in N$ be such that x is analytic for σ_t^φ and $\sigma_z^\varphi(x) \in \mathscr{N}_T$ for all $z \in \mathbb{C}$. Then $\Lambda_T(x)$ is analytic for $\sigma_t^{\varphi_Q}$, with $\sigma_z^{\varphi_Q}(\Lambda_T(x)) = \Lambda_T(\sigma_z^\varphi(x))$ for all $z \in \mathbb{C}$.*

[7] In fact, we will not show that the constructed unitary equals the previous one, as we will not need to know this, but in any case, it is not difficult to prove.

Proof. First remark that $\Lambda_T(x) \in Q_{12}$, for example by [31], Lemma 10.6.(i). Choose $y \in \mathscr{N}_\mu$ and $u, v \in \mathscr{N}_\varphi$ with v in the Tomita algebra $\mathscr{T}_\varphi \subseteq N$ for φ. Denote by ω the normal functional

$$\omega := \omega_{\Lambda_\mu(y), J_N \sigma^\varphi_{i/2}(v) J_N \Lambda_\varphi(u)} \in B(\mathscr{L}^2(N_0), \mathscr{L}^2(N))_*,$$

and denote $f_\omega(z) := \omega(\Lambda_T(\sigma^\varphi_z(x)))$ for $z \in \mathbb{C}$. Then

$$\begin{aligned} f_\omega(z) &= \langle J_N \sigma^\varphi_{i/2}(v)^* J_N \Lambda_\varphi(\sigma^\varphi_z(x) y), \Lambda_\varphi(u)\rangle \\ &= \langle \sigma^\varphi_z(x) \Lambda_\varphi(yv), \Lambda_\varphi(u)\rangle, \end{aligned}$$

and so f_ω is analytic. Moreover, if $z = r + is$ with $r, s \in \mathbb{R}$, then since $\sigma^\mu_t = (\sigma^\varphi_t)_{|N_0}$,

$$\begin{aligned} |f_\omega(z)| &= |\langle \sigma^\varphi_{is}(x) \nabla_\varphi^{-ir} \Lambda_\varphi(yv), \nabla_\varphi^{-ir} \Lambda_\varphi(u)\rangle| \\ &= |\langle \Lambda_\varphi(\sigma^\varphi_{is}(x) \sigma^\mu_{-r}(y)), J_N \sigma^\varphi_{i/2}(\sigma^\varphi_{-r}(v)) J_N \Lambda_\varphi(\sigma^\varphi_{-r}(u))\rangle| \\ &= |\langle \Lambda_T(\sigma^\varphi_{is}(x)) \nabla_\mu^{-ir} \Lambda_\mu(y), \nabla_\varphi^{-ir} J_N \sigma^\varphi_{i/2}(v) J_N \Lambda_\varphi(u)\rangle|, \end{aligned}$$

and so we can conclude, by the Phragmén-Lindelöf principle, that the modulus of f_ω is bounded on every horizontal strip S by $M_{x,S}\|\omega\|$, where $M_{x,S}$ is a number depending only on x and the chosen strip S. The same is of course true for linear combinations of such ω, and since these span a dense subspace of $B(\mathscr{L}^2(N_0), \mathscr{L}^2(N))_*$, we get that $z \to \Lambda_T(\sigma^\varphi_z(x))$ is bounded on compact sets. But then this function is analytic (for example by condition $A.1.(iii)$ in the appendix of [84]). Since $\sigma^{\varphi_Q}_t$ is implemented by $\nabla^{it}_\varphi \oplus \nabla^{it}_\mu$ and $\nabla^{it}_\varphi \Lambda_T(x) \nabla^{-it}_\mu = \Lambda_T(\sigma^\varphi_t(x))$, the result follows.

□

We can now provide a convenient Tomita algebra for φ_2. Let $\mathscr{T}_\varphi \subseteq N$ be the Tomita algebra for φ, and denote

$$\mathscr{T}_{\varphi,T} = \{x \in \mathscr{T}_\varphi \cap \mathscr{N}_T \cap \mathscr{N}_T^* \mid \sigma^\varphi_z(x) \in \mathscr{N}_T \cap \mathscr{N}_T^* \text{ for all } z \in \mathbb{C}\}.$$

(This space is called the Tomita algebra for φ and T in Proposition 2.2.1 of [30].) Denote the linear span of $\{\Lambda_T(x)\Lambda_T(y)^* \mid x, y \in \mathscr{T}_{\varphi,T}\}$ by $\mathfrak{A}_2$, and further denote by $(\mathscr{L}^2(N_2), \Lambda_{\varphi_2}, \pi_{N_2})$ the GNS construction for φ_2.

Proposition 5.7.3. *We have* $\mathfrak{A}_2 \subseteq \mathscr{D}(\Lambda_{\varphi_2})$*, and* $\mathfrak{A}_2$ *is a Tomita algebra for* (N_2, φ_2).

Proof. For $x \in \mathscr{T}_{\varphi,T}$, we know that $\Lambda_T(x)\Lambda_T(x)^* \in \mathscr{M}_{T_2}^+$, with

$$T_2(\Lambda_T(x)\Lambda_T(x)^*) = xx^*.$$

Since $x \in \mathscr{T}_\varphi$, also $xx^* \in \mathscr{M}_\varphi^+$. Hence $\mathfrak{A}_2 \subseteq \mathscr{M}_{\varphi_2}$ by polarization, and so certainly $\mathfrak{A}_2 \subseteq \mathscr{D}(\Lambda_{\varphi_2})$.

It is clear that $\mathfrak{A}_2$ is closed under the *-involution. Now choose $x, y, u, v \in \mathscr{T}_{\varphi,T}$. Then

$$(\Lambda_T(u)\Lambda_T(v)^*)(\Lambda_T(x)\Lambda_T(y)^*) = \Lambda_T(uT(v^*x))\Lambda_T(y)^*.$$

We want to show that $uT(v^*x) \in \mathscr{T}_{\varphi,T}$. It is clear that

$$uT(v^*x) \in \mathscr{N}_\varphi^* \cap \mathscr{N}_T \cap \mathscr{N}_T^*.$$

By the previous lemma, we have, using notation as there, that $\Lambda_T(v)$ and $\Lambda_T(x)$ are analytic for $\sigma_t^{\varphi_Q}$, with

$$\sigma_z^{\varphi_Q}(\Lambda_T(v)) = \Lambda_T(\sigma_z^\varphi(v))$$

and

$$\sigma_z^{\varphi_Q}(\Lambda_T(x)) = \Lambda_T(\sigma_z^\varphi(x))$$

for all $z \in \mathbb{C}$. But then also $\Lambda_T(v)^*\Lambda_T(x) = T(v^*x)$ analytic for $\sigma_t^{\varphi_Q}$, with

$$\sigma_z^{\varphi_Q}(T(v^*x)) = T(\sigma_{\bar{z}}^\varphi(v)^*\sigma_z^\varphi(x))$$

for all $z \in \mathbb{C}$. Since $\sigma_t^{\varphi_Q}$ restricts to σ_t^μ on N_0, and also σ_t^φ restricts to σ_t^μ on N_0, we get that $uT(v^*x)$ is analytic for σ_t^φ, with

$$\sigma_z^\varphi(uT(v^*x)) = \sigma_z^\varphi(u)T(\sigma_{\bar{z}}^\varphi(v)^*\sigma_z^\varphi(x))$$

for $z \in \mathbb{C}$. Since $\mathscr{T}_{\varphi,T}$ is invariant under all σ_z^φ with $z \in \mathbb{C}$, we get that

$$\sigma_z^\varphi(uT(v^*x)) \in \mathscr{N}_\varphi^* \cap \mathscr{N}_T \cap \mathscr{N}_T^*$$

for all $z \in \mathbb{C}$. Hence $uT(v^*x) \in \mathscr{T}_{\varphi,T}$, and thus

$$(\Lambda_T(u)\Lambda_T(v)^*)(\Lambda_T(x)\Lambda_T(y)^*) \in \mathfrak{A}_2.$$

We have shown that $\Lambda_{\varphi_2}(\mathfrak{A}_2)$ is a sub-left Hilbert algebra of $\Lambda_{\varphi_2}(\mathscr{N}_{\varphi_2}\cap\mathscr{N}_{\varphi_2}^*)$. But by the previous lemma, $\mathfrak{A}_2$ consists of analytic elements for $\sigma_t^{\varphi_Q}$, which restricts to $\sigma_t^{\varphi_2}$ on N_2. So in fact $\mathfrak{A}_2$ is a sub*-algebra of $\mathscr{T}_{\varphi_2}$, invariant under the (complex) modular one-parametergroup.

So to end, we have to show that $\mathfrak{A}_2$ is σ-weakly dense in N_2. For this, it is enough to show that $\Lambda_T(\mathscr{T}_{\varphi,T})$ is strongly dense in Q_{12}. Note that $\Lambda_T(\mathscr{T}_{\varphi,T})$ is closed under right multiplication with elements from $\mathscr{T}_\mu \subseteq N_0$, which are σ-weakly dense in N_0. Then by a similar argument as in the proof of Theorem 10.6.(ii), it is sufficient to prove that if $z \in Q_{12}$ and $z^*\Lambda_T(x) = 0$ for all $x \in \mathscr{T}_{\varphi,T}$, then $z = 0$. So suppose z satisfies this condition. Choose $y \in \mathscr{N}_\mu$ analytic for σ_t^μ. Then

$$\begin{aligned}\theta_{N_0}(\sigma_{i/2}^{\mu}(y))z^*\Lambda_\varphi(x) &= z^*\theta_N(\sigma_{i/2}^{\varphi}(y))\Lambda_\varphi(x)\\ &= z^*\Lambda_\varphi(xy)\\ &= z^*\Lambda_T(x)\Lambda_\mu(y)\\ &= 0.\end{aligned}$$

Letting $\theta_{N_0}(\sigma_{i/2}^{\mu}(y))$ tend to 1, we see that z^* vanishes on $\Lambda_\varphi(\mathscr{T}_{\varphi,T})$. Now choose $x \in \mathscr{M}_\varphi \cap \mathscr{M}_T$. Then $x_n = \sqrt{\frac{n}{\pi}}\int_{-\infty}^{+\infty} e^{-nt^2}\sigma_t^\varphi(x)dt$ is in $\mathscr{T}_{\varphi,T}$ by Lemma 10.12 of [31], and $\Lambda_\varphi(x_n)$ converges to $\Lambda_\varphi(x)$. Hence z^* vanishes on $\Lambda_\varphi(\mathscr{M}_\varphi \cap \mathscr{M}_T)$. Since $\mathscr{N}_\varphi \cap \mathscr{N}_T$ is weakly dense in N and $\Lambda_\varphi(\mathscr{N}_\varphi \cap \mathscr{N}_T)$ is normdense in $\mathscr{L}^2(N)$, we get that $z^* = 0$, and the density claim follows.

□

Remark: It also follows easily from Lemma 10.12 of [31] that $\mathscr{T}_{\varphi,T}$ itself is σ-weakly dense in N.

Let $\mathscr{L}^2(N) \underset{\mu}{\otimes} \mathscr{L}^2(N)$ denote the Connes-Sauvageot tensor product, with its natural N_2-N_2-equivalence correspondence structure. Denote by $\mathscr{K}$ the natural image of the algebraic tensor product $\Lambda_\varphi(\mathscr{T}_{\varphi,T}) \odot \Lambda_\varphi(\mathscr{T}_{\varphi,T})$ inside $\mathscr{L}^2(N) \underset{\mu}{\otimes} \mathscr{L}^2(N)$.

Lemma 5.7.4. *For $x, y, z, w \in \mathscr{T}_{\varphi,T}$, we have*

$$\langle \Lambda_\varphi(x) \underset{\mu}{\otimes} \Lambda_\varphi(y), \Lambda_\varphi(z) \underset{\mu}{\otimes} \Lambda_\varphi(w)\rangle = \varphi(w^*T(z^*x)y).$$

Proof. First note that the expression on the left is well-defined by Theorem 10.6.(v) of [31], and then by definition, we have for $x, y, z, w \in \mathscr{T}_{\varphi,T}$ that

$$\begin{aligned}&\langle \Lambda_\varphi(x) \underset{\mu}{\otimes} \Lambda_\varphi(y), \Lambda_\varphi(z) \underset{\mu}{\otimes} \Lambda_\varphi(w)\rangle \\ &= \langle (\Lambda_T(z)^*\Lambda_T(x))\Lambda_\varphi(y), \Lambda_\varphi(w)\rangle \\ &= \varphi(w^*T(z^*x)y).\end{aligned}$$

□

Theorem 5.7.5. *Let $N_0 \subseteq N$ be a normal inclusion of von Neumann algebras, and N_2 its basic construction. Let μ be an nsf weight on N_0, and let T be an nsf N_0-valued weight on N. Let μ, φ, φ_2 be the tower construction for μ w.r.t. T. Then the space $\mathscr{K}$ introduced above is dense in $\mathscr{L}^2(N) \underset{\mu}{\otimes} \mathscr{L}^2(N)$, and the map*

$$\mathscr{K} \to \mathscr{L}^2(N_2) : \Lambda_\varphi(x) \underset{\mu}{\otimes} \Lambda_\varphi(y) \to \Lambda_{\varphi_2}(\Lambda_T(x)\Lambda_T(y^*)^*)$$

extends to a unitary equivalence of N_2-N_2-equivalence correspondences.

Proof. By the previous lemma, we have for $x, y, z, w \in \mathscr{T}_{\varphi,T}$ that

$$\begin{aligned}&\langle \Lambda_\varphi(x) \underset{\mu}{\otimes} \Lambda_\varphi(y), \Lambda_\varphi(z) \underset{\mu}{\otimes} \Lambda_\varphi(w)\rangle \\ &= \varphi(w^*T(z^*x)y) \\ &= \langle \Lambda_{\varphi_2}(\Lambda_T(x)\Lambda_T(y^*)^*), \Lambda_{\varphi_2}(\Lambda_T(z)\Lambda_T(w^*)^*)\rangle,\end{aligned}$$

so that the given map extends to a well-defined partial isometry. Since $\Lambda_\varphi(\mathscr{T}_{\varphi,T})$ is dense in $\mathscr{L}^2(N)$ (which was proven in the course of the previous proposition), we have that $\mathscr{K}$ is dense in $\mathscr{L}^2(N) \underset{\mu}{\otimes} \mathscr{L}^2(N)$. Since also $\Lambda_{\varphi_2}(\mathfrak{A}_2)$ is dense in $\mathscr{L}^2(N_2)$, the extension is in fact a unitary.

The fact that it is a bimodule map follows from a straightforward computation (since we only have to check the bimodule property for operators in $\mathfrak{A}_2$ and vectors in $\mathscr{K}$ and $\Lambda_{\varphi_2}(\mathfrak{A}_2)$).

□

In the following, *we will always identify $\mathscr{L}^2(N) \underset{\mu}{\otimes} \mathscr{L}^2(N)$ and $\mathscr{L}^2(N_2)$ in this manner of the above theorem*, transporting structure from one Hilbert space to the other without any further comment.

Corollary 5.7.6. *If $x, y \in \mathscr{T}_{\varphi,T}$, then*

$$\nabla_{\varphi_2}^{it}(\Lambda_\varphi(x) \underset{\mu}{\otimes} \Lambda_\varphi(y)) = \Lambda_\varphi(\sigma_t^\varphi(x)) \underset{\mu}{\otimes} \Lambda_\varphi(\sigma_t^\varphi(y)).$$

Proof. This follows straightforwardly from the concrete form of the identification of $\mathscr{L}^2(N) \underset{\mu}{\otimes} \mathscr{L}^2(N)$ and $\mathscr{L}^2(N_2)$ given in the previous theorem, using that

$$\sigma_t^{\varphi_2}(\Lambda_T(x)\Lambda_T(y)^*) = \Lambda_T(\sigma_t^\varphi(x))\Lambda_T(\sigma_t^\varphi(y))^*.$$

□

Lemma 5.7.7. *Let x, y be elements of $\mathscr{T}_{\varphi,T}$, and let p be an element of $\mathscr{N}_{\varphi_2}$. Then*

$$\langle \Lambda_\varphi(x) \underset{\mu}{\otimes} \Lambda_\varphi(y), \Lambda_{\varphi_2}(p) \rangle = \langle \Lambda_\varphi(x), p\Lambda_\varphi(\sigma_{-i}^\varphi(y^*)) \rangle.$$

Conversely, if $p \in N_2$ and $\xi \in \mathscr{L}^2(N_2)$ are such that

$$\langle \Lambda_\varphi(x) \underset{\mu}{\otimes} \Lambda_\varphi(y), \xi \rangle = \langle \Lambda_N(x), p\Lambda_\varphi(\sigma_{-i}^\varphi(y^*)) \rangle$$

for all $x, y \in \mathscr{T}_{\varphi,T}$, then $p \in \mathscr{N}_{\varphi_2}$ and $\Lambda_{\varphi_2}(p) = \xi$.

Proof. Suppose $p = \Lambda_T(z)\Lambda_T(w^*)^*$ for some $z, w \in \mathscr{T}_{\varphi,T}$. Then since $w^*T(z^*x) \in \mathscr{N}_\varphi \cap \mathscr{N}_\varphi^*$, we have

$$\begin{aligned}
\langle \Lambda_\varphi(x) \underset{\mu}{\otimes} \Lambda_\varphi(y), \Lambda_{\varphi_2}(p) \rangle &= \langle \Lambda_\varphi(x) \underset{\mu}{\otimes} \Lambda_\varphi(y), \Lambda_\varphi(z) \underset{\mu}{\otimes} \Lambda_\varphi(w) \rangle \\
&= \varphi(w^*T(z^*x)y) \\
&= \varphi(\sigma_i^\varphi(y)w^*T(z^*x)) \\
&= \langle \Lambda_\varphi(w^*T(z^*x)), \Lambda_\varphi(\sigma_{-i}^\varphi(y^*)) \rangle \\
&= \langle \Lambda_T(w^*)\Lambda_T(z)^*\Lambda_\varphi(x), \Lambda_\varphi(\sigma_{-i}^\varphi(y^*)) \rangle \\
&= \langle \Lambda_\varphi(x), p\Lambda_\varphi(\sigma_{-i}^\varphi(y^*)) \rangle.
\end{aligned}$$

As $\mathfrak{A}_2$, being a Tomita algebra for φ_2, is a σ-strong-norm core for Λ_{φ_2}, the result holds true for any $p \in \mathscr{N}_{\varphi_2}$.

Now we prove the converse statement. So let $p \in N_2$ and $\xi \in \mathscr{L}^2(N_2)$ be such that

$$\langle \Lambda_\varphi(x) \underset{\mu}{\otimes} \Lambda_\varphi(y), \xi \rangle = \langle \Lambda_\varphi(x), p\Lambda_\varphi(\sigma_{-i}^\varphi(y^*)) \rangle$$

for all $x, y \in \mathscr{T}_{\varphi,T}$. Then, since $\mathfrak{A}_2$ is also a σ-strong-norm core for $\Lambda^{\mathrm{op}}_{\varphi_2}$, it is enough to prove that $p\Lambda^{\mathrm{op}}_{\varphi_2}(a) = \theta_{N_2}(a)\xi$ for all $a \in \mathfrak{A}_2$, by Proposition 5.3.6. Now if $a = \Lambda_T(x)\Lambda_T(y^*)^*$, then $a \in \mathscr{T}_{\varphi_2}$ with

$$\begin{aligned}\Lambda^{\mathrm{op}}_{\varphi_2}(a) &= J_{N_2}\Lambda_{\varphi_2}(a^*)\\ &= \Lambda_{\varphi_2}(\sigma^{\varphi_2}_{-i/2}(a))\\ &= \Lambda_{\varphi_2}(\Lambda_T(\sigma^{\varphi}_{-i/2}(x))\Lambda_T(\sigma^{\varphi}_{-i/2}(y)^*)^*).\end{aligned}$$

So if also $b \in \mathfrak{A}_2$ with $b = \Lambda_T(z)\Lambda_T(w^*)^*$, $w, z \in \mathscr{T}_{\varphi,T}$, then

$$\begin{aligned}\langle \Lambda_{\varphi_2}(b), p\Lambda^{\mathrm{op}}_{\varphi_2}(a)\rangle &= \langle \Lambda_{\varphi_2}(b), \Lambda_{\varphi_2}(p\Lambda_T(\sigma^{\varphi}_{-i/2}(x))\Lambda_T(\sigma^{\varphi}_{-i/2}(y)^*)^*)\\ &= \langle \Lambda_{\varphi}(z), p\Lambda_T(\sigma^{\varphi}_{-i/2}(x))\Lambda_T(\sigma^{\varphi}_{-i/2}(y)^*)^*\Lambda_{\varphi}(\sigma^{\varphi}_{-i}(w^*))\rangle\end{aligned}$$

by the first part of the lemma. On the other hand, we have

$$\begin{aligned}\langle \Lambda_{\varphi_2}(b), \theta_{N_2}(a)\xi\rangle &= \langle \theta_{N_2}(a)^*\Lambda_{\varphi_2}(b), \xi\rangle\\ &= \langle \Lambda_{\varphi_2}(b\sigma^{\varphi_2}_{i/2}(a)^*), \xi\rangle\\ &= \langle \Lambda_{\varphi_2}(\Lambda_T(z)\Lambda_T(w^*)^*\Lambda_T(\sigma^{\varphi}_{i/2}(y)^*)\Lambda_T(\sigma^{\varphi}_{i/2}(x))^*), \xi\rangle\\ &= \langle \Lambda_{\varphi_2}(\Lambda_T(z)\Lambda_T(\sigma^{\varphi}_{i/2}(x)T(\sigma^{\varphi}_{i/2}(y)w^*))^*), \xi\rangle\\ &= \langle \Lambda_{\varphi}(z), p\Lambda_{\varphi}(\sigma^{\varphi}_{-i/2}(x)T(\sigma^{\varphi}_{-i/2}(y)\sigma^{\varphi}_{-i}(w^*)))\rangle,\end{aligned}$$

the last step by our assumption. Since this equals our earlier expression, we have proven that $p\Lambda^{\mathrm{op}}_{\varphi_2}(a) = \theta_{N_2}(a)\xi$ for all $a \in \mathfrak{A}_2$.

□

We prove three further results which naturally belong here.

Lemma 5.7.8. *Let $N_0 \subseteq N$ be a unital normal inclusion of von Neumann algebras, T an nsf operator valued weight from N onto N_0, μ an nsf weight on N_0, and φ the nsf weight $\mu \circ T$. Suppose $x \in N$ and $z \in B(\mathscr{L}^2(N_0), \mathscr{L}^2(N))$ are such, that for any $y \in \mathscr{N}_\mu$, we have $xy \in \mathscr{N}_\varphi$ and $\Lambda_\varphi(xy) = z\Lambda_\mu(y)$. Then $x \in \mathscr{N}_T$ with $\Lambda_T(x) = z$.*

Proof. Choose $y, w \in \mathscr{N}_\mu$ with w in the Tomita algebra of μ. Then

$$\begin{aligned}\theta_N(w)z\Lambda_\mu(y) &= \theta_N(w)\Lambda_\varphi(xy)\\ &= \Lambda_\varphi(xy\sigma^{\varphi}_{-i/2}(w))\\ &= z\Lambda_\mu(y\sigma^{\mu}_{-i/2}(w))\\ &= z\theta_{N_0}(w)\Lambda_\mu(y),\end{aligned}$$

so that z is a right N_0-module map. It follows that $z^*z \in N_0$.

Now for any element $u \in N_0^{+,\mathrm{ext}}$, one can find a sequence $u_n \in N_0^+$ such that $u_n \nearrow u$ pointwise on $(N_0)_*^+$ (see the proof of Proposition 4.17.(ii) in [84]). From this, it follows that for every $y \in \mathscr{N}_\mu$,

$$\omega_{\Lambda_\mu(y),\Lambda_\mu(y)}(T(x^*x)) = \mu(y^*T(x^*x)y),$$

using Corollary 4.9 of [84] (which allows us to extend weights to the extended positive cone). Using the bimodularity of T, the right hand side equals $\varphi((xy)^*(xy) = (\mu \circ T)(y^*x^*xy)$, which is bounded by assumption. Since this last expression equals $\langle z\Lambda_\mu(y), z\Lambda_\mu(y)\rangle$, again by assumption, we see that

$$\omega_{\Lambda_\mu(y),\Lambda_\mu(y)}(T(x^*x)) = \omega_{\Lambda_\mu(y),\Lambda_\mu(y)}(z^*z)$$

for all $y \in \mathscr{N}_\mu$. By the lower-semi-continuity of T, we conclude that $T(x^*x)$ is bounded, and then of course $\Lambda_T(x) = z$ follows.

□

Lemma 5.7.9. *Let* $\begin{matrix} N_{10} & \subseteq & N_{11} \\ \cup\!| & & \cup\!| \\ N_{00} & \subseteq & N_{01} \end{matrix}$ *be unital normal inclusions of von Neumann algebras. Denote, for $i \in \{0,1\}$, by Q_i the linking algebra between the right N_{i0}-modules $\mathscr{L}^2(N_{i0})$ and $\mathscr{L}^2(N_{i1})$. Suppose T_1 is an nsf operator valued weight $N_{11}^+ \to N_{10}^{+,ext}$ whose restriction to N_{01}^+ determines an nsf operator valued weight $N_{01}^+ \to N_{00}^{+,ext}$, in the sense that $\omega(T_0(x)) = \omega(T_1(x))$ for all $\omega \in (N_{10})_*^+$ and $x \in N_{01}^+$. Then there is a natural normal embedding of Q_0 into Q_1, determined by $\Lambda_{T_0}(x) \to \Lambda_{T_1}(x)$ for $x \in \mathscr{N}_{T_0}$.*

Remark: The inclusion will in general *not* be unital. Consider for example the case where $N_{11} = M_2(\mathbb{C})$ and all other algebras equal to $\mathbb{C}$.

Proof. By assumption, if $x, y \in \mathscr{N}_{T_0}$, then $x, y \in \mathscr{N}_{T_1}$, and $T_0(x^*y) = T_1(x^*y)$. Denote by $\tilde{\mathscr{Q}}_1$ the *-algebra generated by the $\Lambda_{T_1}(x)$, $x \in \mathscr{N}_{T_0}$, and by $\tilde{Q}_1$ its σ-weak closure inside Q_1. Denote by $\mathscr{Q}_0$ the *-algebra generated by the $\Lambda_{T_0}(x)$, $x \in \mathscr{N}_{T_0}$. We want to show that Q_0 and $\tilde{Q}_1$ are isomorphic in the indicated way.

Now for $a_i, b_i \in \mathscr{N}_{T_0}$, it is easy to check that $\sum_i \Lambda_{T_1}(a_i)\Lambda_{T_1}(b_i)^* = 0$ iff $\sum_i \Lambda_{T_0}(a_i)\Lambda_{T_0}(b_i)^* = 0$, so we already have an isomorphism F at the level of $\mathscr{Q}_0$ and $\tilde{\mathscr{Q}}_1$. Denote by e_0 the unit of N_{00}, seen as a projection in Q_0, and denote by e_1 the unit of N_{00} as a projection in $\tilde{Q}_1$. Suppose that x_i is a bounded net in $\mathscr{Q}_0$ which converges to 0 in the σ-weak topology. Then for any $a,b \in \mathscr{Q}_0$, we have that $e_0 a x_i b e_0$ converges to 0 σ-weakly. Applying F, we get that $e_1 F(a)F(x_i)F(b)e_1$ converges σ-weakly to 0, and then also $ce_1F(a)F(x_i)F(b)e_1 d$, for any $c,d \in \tilde{Q}_1$. Since $\tilde{Q}_1 e_1 \tilde{\mathscr{Q}}_1$ is σ-weakly dense in $\tilde{Q}_1$, we get that $F(x_i)$ converges σ-weakly to 0. Since the same argument applies to F^{-1}, we see that F extends to a *-isomorphism between Q_0 and $\tilde{Q}_1$, and we are done. □

Remark: We could also have used the results from [66] concerning self-dual Hilbert W^*-modules.

When $\mathscr{H}$ is a Hilbert space, and A a (possibly unbounded) positive operator on $\mathscr{H}$, we denote by $\mathrm{Tr}(\,\cdot A)$ the nsf weight on $B(\mathscr{H})$ such that, with ξ_i denoting an orthonormal basis of $\mathscr{H}$ consisting of vectors in $\mathscr{D}(A^{1/2})$, we have

$$\mathrm{Tr}(\,\cdot A)(x) = \sum_i \|x^{1/2}A^{1/2}\xi_i\|^2 \qquad \text{for } x \in B(\mathscr{H})^+.$$

Its modular one-parametergroup is implemented on $\mathscr{H}$ by A^{it}.

Lemma 5.7.10. *Let $\mathscr{H}$ be a Hilbert space, and φ an nsf weight on $B(\mathscr{H})$. Let Tr be the canonical trace on $B(\mathscr{H})$, and A the positive, densily defined operator such that $\varphi = Tr(\,\cdot A)$. Then, under the canonical identification $B(\mathscr{H}) \otimes B(\overline{\mathscr{H}}) \to B(\mathscr{L}^2(B(\mathscr{H}))$, the operator valued weight $T_\varphi : B(\mathscr{L}^2(B(\mathscr{H}))) \to B(\mathscr{H})$, obtained from the inclusion $\mathbb{C} \underset{\varphi}{\subseteq} B(\mathscr{H})$, corresponds to the operator valued weight $\iota \otimes Tr(\,\cdot \overline{A}^{-1})$.*

Proof. Note that $\mathscr{H} \otimes \overline{\mathscr{H}}$ can be identified with $\mathscr{L}^2(B(\mathscr{H}))$ by sending $\xi \otimes \overline{\eta}$ to $\Lambda_{\mathrm{Tr}}(l_\xi l_\eta^*)$, by which we identify $B(\mathscr{H}) \otimes B(\overline{\mathscr{H}})$ with $B(\mathscr{L}^2(B(\mathscr{H})))$. We explicitly denote this map by Φ.

Now $\varphi_2 := \varphi \circ T_\varphi$ will equal $\mathrm{Tr}(\,\cdot \nabla_\varphi)$, since $\frac{d\varphi_2}{d\mu'} = \nabla_\varphi$, where μ is just the identity map on $\mathbb{C}$. Moreover, it is well-known (and easy to establish) that

$$\Phi^{-1}(\nabla_\varphi^{it}) = A^{it} \otimes \overline{A}^{it}.$$

Hence

$$(\varphi \circ T_\varphi) \circ \Phi = \mathrm{Tr}(\,\cdot A) \otimes \mathrm{Tr}(\,\cdot \overline{A}^{-1}).$$

Clearly, $\tilde{T}_\varphi := \Phi \circ (\iota \otimes \mathrm{Tr}(\,\cdot \overline{A}^{-1}) \circ \Phi^{-1}$ is an nsf operator valued weight satisfying $\varphi \circ \tilde{T}_\varphi = \mathrm{Tr}(\,\cdot \nabla_\varphi)$. By uniqueness (Theorem IX.4.18 of [84]), $\tilde{T}_\varphi = T_\varphi$.

□

Chapter 6

Preliminaries on locally compact quantum groups

In this chapter, we recall the main results from [56], [57] and [85] on von Neumann algebraic and C*-algebraic quantum groups and their coactions. We also develop some *new* results concerning integrable coactions in the fourth section.

6.1 von Neumann algebraic quantum groups

Definition 6.1.1. *A* Hopf-von Neumann algebra[1] *is a couple* (M, Δ_M) *consisting of a von Neumann algebra* M *and a unital normal faithful* **-homomorphism* $\Delta_M : M \to M \otimes M$*, called the* coproduct *or* comultiplication*, such that*

$$(\Delta_M \otimes \iota_M)\Delta_M = (\iota_M \otimes \Delta_M)\Delta_M \qquad \textit{(coassociativity)}.$$

A Hopf-von Neumann algebra is called coinvolutive *if there exists an involutive anti-***-automorphism* $R_M : M \to M$ *such that*

$$\Delta_M \circ R_M = (R_M \otimes R_M) \circ \Delta_M^{op}.$$

Such an R_M *is then called* a coinvolution.

[1] The terminology von Neumann bialgebra would be better suited, but we will keep the terminology as it is used in the literature

As for Hopf algebras, we will simply denote a Hopf-von Neumann algebra by the symbol for its underlying von Neumann algebra.

The following object was studied in [57] (see also [95]).

Definition 6.1.2. A von Neumann algebraic quantum group *is a Hopf-von Neumann algebra M for which there exist nsf weights φ_M and ψ_M on the von Neumann algebra M, such that for all non-zero $\omega \in (M_*)^+$, we have, for $x \in \mathscr{M}_{\varphi_M}^+$,*

$$\varphi_M((\omega \otimes \iota_M)\Delta_M(x)) = \omega(1)\varphi_M(x) \qquad \textit{(left invariance)},$$

and, for $x \in \mathscr{M}_{\psi_M}^+$,

$$\psi_M((\iota_M \otimes \omega)\Delta_M(x)) = \omega(1)\psi_M(x) \qquad \textit{(right invariance)}.$$

In [57], it is then proven that these invariance properties imply the following stronger statement.

Lemma 6.1.3. *Let M be a von Neumann algebraic quantum group and N a von Neumann algebra. Then for $\omega \in (N \otimes M)_*^+$ and $x \in (N \otimes M)^+$, we have*

$$\omega((\iota_N \otimes \iota_M \otimes \varphi_M)((\iota_N \otimes \Delta_M)(x))) = \omega_1((\iota_N \otimes \varphi_M)(x))$$

where $\omega_1(x) := \omega(x \otimes 1_M)$ for $x \in N$. Similarly for ψ_M.

The previous lemma implies in particular that for any non-zero positive functional ω on a von Neumann algebraic quantum group, the nsf weight $(\omega \otimes \varphi_M)\Delta_M$ on M equals the nsf weight $\omega(1_M)\varphi_M$. It also implies that $\mathscr{N}_{(\iota\otimes\varphi_M)} \cap \Delta_M(M) = \mathscr{N}_{\varphi_M}$.

von Neumann algebraic quantum groups have a lot of extra structure, which would maybe not be expected given this airy definition. We recall some of the most important results. They are however not ordered in the way one should prove them!

Proposition 6.1.4. *Let M be a von Neumann algebraic quantum group. If φ_M and $\tilde{\varphi}_M$ are left invariant nsf weights, then there exists $r \in \mathbb{R}_0^+$ with $\tilde{\varphi}_M = r \cdot \varphi_M$. Similarly, all right invariant nsf weights are scalar multiples of each other.*

In the following, we will always suppose that we have associated some fixed left invariant nsf weight with a von Neumann algebraic quantum group M. By the following results, once this left invariant weight is fixed, one can *canonically* associate to it a right invariant weight.

Definition-Proposition 6.1.5. *Let M be a von Neumann algebraic quantum group. There exists a unique couple (τ_t^M, R_M), consisting of a one-parametergroup of *-automorphisms τ_t^M of M and an involutive anti-*-automorphism R_M of M, such that $R_M \circ \tau_t^M = \tau_t^M \circ R_M$, and such that, with $S_M = R_M \circ \tau_{-i/2}^M$, we have, for $x, y \in \mathscr{N}_{\varphi_M}$, that $(\iota_M \otimes \varphi_M)(\Delta_M(y)^*(1 \otimes x)) \in \mathscr{D}(S_M)$, with*

$$S_M((\iota_M \otimes \varphi_M)(\Delta_M(y)^*(1 \otimes x)))^* = (\iota_M \otimes \varphi_M)(\Delta_M(x)^*(1 \otimes y)).$$

This property is called strong left invariance.

The one-parametergroup τ_t^M is called the scaling group *of M. The anti-automorphism R_M is called the* unitary antipode *of M. The map S_M is called* the antipode *of M.*

We have that τ_t^M commutes with $\sigma_s^{\varphi_M}$ and $\sigma_s^{\psi_M}$ for all $s, t \in \mathbb{R}$, while

$$R_M \circ \sigma_t^{\varphi_M} = \sigma_{-t}^{\psi_M} \circ R_M.$$

Note that if $x, y \in \mathscr{N}_{\varphi_M}$, then $\Delta_M(y)$ and $(1 \otimes x)$ are both in $\mathscr{N}_{(\iota \otimes \varphi_M)}$, so that $(\iota_M \otimes \varphi_M)(\Delta_M(y)^*(1 \otimes x))$ makes sense.

Proposition 6.1.6. *Let M be a von Neumann algebraic quantum group. Then each automorphism τ_t^M of the scaling group is an automorphism of the von Neumann algebraic quantum group M:*

$$\Delta_M \circ \tau_t^M = (\tau_t^M \otimes \tau_t^M) \circ \Delta_M.$$

On the other hand, the unitary antipode R_M is a coinvolution.

In particular, if φ_M is a left invariant nsf weight, then $\psi_M := \varphi_M \circ R_M$ is a right invariant nsf weight.

As said, we will then always suppose that a left invariant nsf weight φ_M has been fixed, and will take $\psi_M := \varphi_M \circ R_M$ as the right invariant nsf weight.

Definition-Proposition 6.1.7. *Let M be a von Neumann algebraic quantum group. Then there exists a number $\nu_M \in \mathbb{R}_0^+$, called the* scaling constant*, such that $\varphi_M \circ \tau_t^M = \nu_M^t \varphi_M$ for $t \in \mathbb{R}$.*

This allows us to construct a canonical unitary implementation for τ_t^M: we denote by P_M^{it} the unique unitary on $\mathscr{L}^2(M)$ for which

$$P_M^{it}\Lambda_{\varphi_M}(x) = \nu_M^{-t/2}\Lambda_{\varphi_M}(\tau_t^M(x)), \qquad x \in \mathscr{N}_{\varphi_M}.$$

There is a further strong connection between φ_M and ψ_M: the unitary 1-cocycle relating them is almost a one-parametergroup.

Definition-Proposition 6.1.8. *Let M be a von Neumann algebraic quantum group. Then there exists a (possibly unbounded) positive operator δ_M affiliated with M (i.e. $\delta_M^{it} \in M$ for all t), called* the modular element*, such that the cocycle derivative of ψ_M w.r.t. φ_M equals $u_t = \nu_M^{it^2/2}\delta_M^{it}$. This implies that $\sigma_t^{\varphi_M}(\delta_M^{is}) = \nu_M^{ist}\delta_M^{is}$ for all $s, t \in \mathbb{R}$.*

Moreover, the δ_M^{it} are group-like elements*:*

$$\Delta_M(\delta_M^{it}) = \delta_M^{it} \otimes \delta_M^{it},$$

which implies that $\tau_t^M(\delta_M^{is}) = \delta_M^{is}$ and $R_M(\delta_M^{it}) = \delta_M^{-it}$ for all $s, t \in \mathbb{R}$.

Proposition 6.1.9. *Let M be a von Neumann algebraic quantum group. Then the GNS map for ψ_M equals the σ-strong-norm closure of the map*

$$\mathscr{N}_{\varphi_M}^{\delta_M} \to \mathscr{L}^2(M) : x \to \nu_M^{-i/8}\Lambda_{\varphi_M}(x\delta_M^{1/2}),$$

where $\mathscr{N}_{\varphi_M}^{\delta_M}$ is the subset of M consisting of left multipliers x for $\delta_M^{1/2}$, for which $x\delta_M^{1/2} \in \mathscr{N}_{\varphi_M}$.

By this last corollary, one may intuitively write $\psi_M = \varphi_M(\delta_M^{1/2} \cdot \delta_M^{1/2})$.

To keep the scaling constant ν_M from popping up at unwanted places, we can, as in the original paper [57], scale the semi-cyclic representation for ψ_M: we write

$$\Gamma_M := \nu_M^{i/8} \cdot \Lambda_{\psi_M}.$$

However, we will still use Λ_{ψ_M} as the fixed GNS construction to transport structure from $\mathscr{L}^2(M, \psi_M)$ to $\mathscr{L}^2(M)$: the only thing which would change if we would use Γ_M instead, is that the modular conjugation would get scaled by a factor $\nu^{i/8}$ (so, with obvious notation, $J_{\Gamma_M} = \nu_M^{i/8} J_{\Lambda_{\psi_M}}$).

We will also write

$$\Lambda_M := \Lambda_{\varphi_M},$$

since, as stated, we will always assume that there is a *fixed* left invariant nsf weight associated with a von Neumann algebraic quantum group. We further write the modular one-parametergroup $\sigma_t^{\varphi_M}$ as σ_t^M, and we write $\sigma_t^{\psi_M}$ as $\sigma_t'^M$. We follow the same convention for the modular operators.

The following definition introduces the notion of a multiplicative unitary. This concept, whose origins go back to Stinespring, was studied in full generality in the influential paper [4].

Definition 6.1.10. *Let $\mathscr{H}$ be a Hilbert space. A unitary $W \in B(\mathscr{H} \otimes \mathscr{H})$ is called a* multiplicative unitary *if W satisfies the* pentagonal identity*:*

$$W_{12}W_{13}W_{23} = W_{23}W_{12}.$$

Definition-Proposition 6.1.11. *Let M be a von Neumann algebraic quantum group. Then for each $x \in \mathscr{N}_{\varphi_M}$ and $\omega \in M_*$, also $(\omega \otimes \iota_M)\Delta_M(x) \in \mathscr{N}_{\varphi_M}$, and there exists a unique unitary $W_M \in M \otimes B(\mathscr{L}^2(M))$ such that*

$$(\omega \otimes \iota)(W_M^*)\Lambda_M(x) = \Lambda_M((\omega \otimes \iota_M)\Delta_M(x))$$

for all such x and ω.

Then W_M is a multiplicative unitary on $\mathscr{L}^2(M) \otimes \mathscr{L}^2(M)$, called the left regular corepresentation*. Moreover, the set*

$$\{(\iota_M \otimes \omega)(W_M) \mid \omega \in B(\mathscr{L}^2(M))\}$$

is σ-weakly dense in M.

In the previous definition-proposition, the existence of W_M^* *as an isometry* is in fact not so difficult to prove. The hard part consists in showing that it is surjective.

We have a similar result on the right.

Definition-Proposition 6.1.12. *Let M be a von Neumann algebraic quantum group. Then for each $x \in \mathscr{N}_{\psi_M}$ and $\omega \in M_*$, we have $(\iota_M \otimes \omega)\Delta_M(x) \in \mathscr{N}_{\psi_M}$, and there exists a unique unitary $V_M \in B(\mathscr{L}^2(M)) \otimes M$ for which*

$$(\iota \otimes \omega)(V_M)\Lambda_{\psi_M}(x) = \Lambda_{\psi_M}((\iota_M \otimes \omega)\Delta_M(x)).$$

Then V_M is a multiplicative unitary on $\mathscr{L}^2(M) \otimes \mathscr{L}^2(M)$, called the right regular corepresentation*. Moreover, the set*

$$\{(\omega \otimes \iota_M)(V_M) \mid \omega \in B(\mathscr{L}^2(M))\}$$

is σ-weakly dense in M.

As the name suggests, the regular corepresentations are specific examples of (unitary) corepresentations.

Definition 6.1.13. *Let M be a von Neumann algebraic quantum group. A* unitary left corepresentation *U of M consists of a Hilbert space $\mathscr{H}$ together with a unitary $U \in M \otimes B(\mathscr{H})$ such that*

$$(\Delta_M \otimes \iota_{B(\mathscr{H})})(U) = U_{13}U_{23}.$$

A unitary right corepresentation *U of M consists of a Hilbert space $\mathscr{H}$ together with a unitary $U \in B(\mathscr{H}) \otimes M$ such that*

$$(\iota_{B(\mathscr{H})} \otimes \Delta_M)(U) = U_{12}U_{13}.$$

The left regular corepresentation of a von Neumann algebraic quantum group can be used to give a nice formula for its antipode.

Proposition 6.1.14. *Let M be a von Neumann algebraic quantum group, with left regular corepresentation W_M and antipode S_M. Then for each $\omega \in B(\mathscr{L}^2(M))_*$, we have $(\iota_M \otimes \omega)(W_M) \in \mathscr{D}(S_M)$, and*

$$S_M((\iota_M \otimes \omega)(W_M)) = (\iota_M \otimes \omega)(W_M^*).$$

The multiplicative unitaries are the key to the duality theory for von Neumann algebraic quantum groups.

Definition-Proposition 6.1.15. *Let M be a von Neumann algebra, W_M the left regular corepresentation. Then the σ-weak closure of*

$$\{(\omega \otimes \iota)(W_M) \mid \omega \in M_*\}$$

is a von Neumann algebra $\widehat{M}$.

For $x \in \widehat{M}$, we have $W_M(x \otimes 1)W_M^* \in \widehat{M} \otimes \widehat{M}$, and

$$\Delta_{\widehat{M}} : \widehat{M} \to \widehat{M} \otimes \widehat{M} : x \to \Sigma W_M(x \otimes 1)W_M^*\Sigma$$

makes $(\widehat{M}, \Delta_{\widehat{M}})$ into a von Neumann algebraic quantum group, called the dual von Neumann algebraic quantum group of M.

The following Proposition shows how the left invariant weight on this dual is defined, and at the same time provides us with a Fourier transform.

Proposition 6.1.16. *Let M be a von Neumann algebraic quantum group. Define $\mathscr{I}_M$ as the set of $\omega \in M_*$ for which the map*

$$\Lambda_M(\mathscr{N}_M) \to \mathbb{C} : \Lambda_{\varphi_M}(x) \to \overline{\omega(x^*)} = \overline{\omega}(x)$$

extends to a bounded functional on $\mathscr{L}^2(M)$, which will then be of the form $\omega_{\xi_\omega} = \langle \cdot, \xi_\omega \rangle$ for a uniquely determined ξ_ω. Further denote by λ_M the (faithful) map

$$\lambda_M : M_* \to \widehat{M} : \omega \to (\omega \otimes \iota_{\widehat{M}})(W_M).$$

Then there exists a unique nsf weight $\varphi_{\widehat{M}}$ on $\widehat{M}$ such that the σ-strong-norm closure of the map

$$\lambda_M(\mathscr{I}) \to \mathscr{L}^2(M) : \lambda_M(\omega) \to \xi_\omega$$

determines a semicyclic representation for $\varphi_{\widehat{M}}$. Moreover, $\varphi_{\widehat{M}}$ will then be a left invariant nsf weight for $\widehat{M}$.

Thus this determines canonically a unitary intertwiner $\mathscr{L}^2(\widehat{M}) \to \mathscr{L}^2(M)$ of left $\widehat{M}$-representations, and we will then transport all structure of $\mathscr{L}^2(\widehat{M})$ to $\mathscr{L}^2(M)$ without further comment. We will then use several notations for the associated semi-cyclic representation, namely $\Lambda_{\varphi_{\widehat{M}}}$, $\Lambda_{\widehat{M}}$ and $\widehat{\Lambda}_M$.

The following proposition states that 'taking the dual' is an involutive operation.

Proposition 6.1.17. *Let M be a von Neumann algebraic quantum group. Then $W_{\widehat{M}} = \Sigma W_M^* \Sigma$, and hence the dual of $\widehat{M}$ coincides with M as a von Neumann algebraic quantum group. Moreover, if $\varphi_{\widehat{M}}$ is the left invariant weight on $\widehat{M}$ constructed from φ_M as in the previous proposition, then the construction of the previous proposition, applied to $\varphi_{\widehat{M}}$, gives us back φ_M.*

We will then always use this constructed weight $\varphi_{\widehat{M}}$ as the fixed left invariant weight on $\widehat{M}$. By the previous proposition, it also follows that for example $\widehat{\Lambda}_{\widehat{M}} = \Lambda_M$.

Apart from the dual, there are some other new von Neumann algebraic quantum groups which can easily be built from a given von Neumann algebraic quantum group. We list them here, together with their left regular corepresentations.

Proposition 6.1.18. *Let M be a von Neumann algebraic quantum group.*

The commutant *von Neumann algebraic quantum group M' is a von Neumann algebraic quantum group with underlying von Neumann algebra M', and coproduct*

$$\Delta_{M'}(x) := (C_M \otimes C_M)\Delta_M(C_M^{-1}(x)).$$

We choose $\varphi_M \circ C_M^{-1}$ as its fixed left invariant nsf weight. Its left regular corepresentation is

$$W_{M'} = (J_M \otimes J_M)W_M(J_M \otimes J_M),$$

which can also be written

$$W_{M'} = (Ad(J_M J_{\widehat{M}}) \otimes \iota_{\widehat{M}})(W_M^*).$$

The co-opposite *von Neumann algebraic quantum group M^{cop} has M as its underlying von Neumann algebra, but coproduct*

$$\Delta_{M^{cop}}(x) = \Delta_M^{op}(x).$$

We choose ψ_M as its fixed left invariant nsf weight. Its left regular corepresentation is

$$W_{M^{cop}} = \Sigma V_M^* \Sigma.$$

There are various relations between the operations of taking 'duals', 'commutants' and 'co-opposites'. We will only need one of them.

Proposition 6.1.19. *Let M be a von Neumann algebraic quantum group. Then the von Neumann algebraic quantum groups $\widehat{M}'$ and $\widehat{M^{cop}}$ are isomorphic by the identity map.*[2] *Moreover, this isomorphism respects the canonical semi-cyclic representations into $\mathscr{L}^2(M)$. The common left regular corepresentation of these von Neumann algebraic quantum groups is V_M.*

[2]To avoid a possible ambiguity: we will always mean $(\widehat{M})'$ by this notation.

It is useful to scale the natural semi-cyclic representation of $\widehat{M}'$ by $\nu^{i/8}$, for which we introduce the following notation:

$$\widehat{\Gamma}_M := \nu^{i/8} \cdot \Lambda_{\widehat{M}'}.$$

Lemma 6.1.20. *Let M be a von Neumann algebraic quantum group. Denote by M_*^δ the space of elements ω in M_* for which $x \to \overline{\omega}(x\delta_M^{-1/2})$ extends from the space of left multipliers of $\delta_M^{-1/2}$ to a normal functional $\overline{\omega}_\delta$ on M. Then $\lambda_{M^{cop}}(M_*^\delta) \cap \mathscr{N}_{\varphi_{\widehat{M}'}}$ is a σ-strong-norm core for $\widehat{\Gamma}_M$, and moreover*

$$\langle \Lambda_M(x), \widehat{\Gamma}_M(m)\rangle = \overline{\omega}_\delta(x) \qquad \textit{for } x \in \mathscr{N}_{\varphi_M}$$

if $m = \lambda_{M^{cop}}(\omega)$.

Proof. Let $x \in \mathscr{N}_{\psi_M}$. Then since

$$\lambda_{M^{\mathrm{cop}}}(M_*) \cap \mathscr{N}_{\varphi_{\widehat{M^{\mathrm{cop}}}}} = \lambda_{M^{\mathrm{cop}}}(M_*)(\mathscr{I}_{M^{\mathrm{cop}}}),$$

by Remark 8.31 of [56], we have for $\omega \in M_*^\delta$ and $x \in \mathscr{N}_{\psi_M}$, and writing $m = \lambda_{M^{\mathrm{cop}}}(\omega)$, that

$$\langle \Lambda_{M^{\mathrm{cop}}}(x), \Lambda_{\widehat{M^{\mathrm{cop}}}}(m)\rangle = \overline{\omega(x^*)},$$

by definition. So if $x \in M$ is a left multiplier of $\delta_M^{1/2}$, and $x\delta_M^{1/2} \in \mathscr{N}_{\varphi_M}$, this becomes

$$\langle \Lambda_M(x\delta_M^{1/2}), \widehat{\Gamma}_M(m)\rangle = \overline{\omega}(x).$$

Then also, if $x \in \mathscr{N}_{\varphi_M}$ is a left multiplier of $\delta_M^{-1/2}$, we have $\langle \Lambda_M(x), \widehat{\Gamma}_M(m)\rangle = \overline{\omega}_\delta(x)$. Since such x form a σ-strong-norm core for Λ_M, and since the $\overline{\omega}_\delta$ is of the form $\omega_{\xi,\eta}$ for certain $\xi, \eta \in \mathscr{L}^2(M)$, we find that this identity holds for all $x \in \mathscr{N}_{\varphi_M}$. This proves the second part of the lemma.

As for the first part, take $\omega \in \mathscr{I}_{M^{\mathrm{cop}}}$. Define a normal functional $\omega_n \in M_*$ by the formula

$$\omega_n(x) = \sqrt{\frac{n}{\pi}} \int_{-\infty}^{+\infty} e^{-nt^2} \omega(\delta_M^{-it} x) dt$$

for $x \in M$. Then $\omega_n \in M_*^\delta$ and $\omega_n \to \omega$ in norm, so that also $\lambda_{M^{\mathrm{cop}}}(\omega_n)$ converges to $\lambda_{M^{\mathrm{cop}}}(\omega_n)$. Moreover, if $x \in \mathscr{N}_{\psi_M}$, we have

$$\begin{aligned}
\overline{\omega_n}(x) &= \sqrt{\frac{n}{\pi}}\int_{-\infty}^{+\infty} e^{-nt^2}\overline{\omega}(x\delta_M^{it})\rangle dt \\
&= \sqrt{\frac{n}{\pi}}\int_{-\infty}^{+\infty} e^{-nt^2}\langle \Lambda_{\psi_M}(x\delta_M^{it}), \Lambda_{\widehat{M^{\mathrm{cop}}}}(\lambda_{M^{\mathrm{cop}}}(\omega))\rangle dt \\
&= \sqrt{\frac{n}{\pi}}\int_{-\infty}^{+\infty} e^{-nt^2}\nu^{-t/2}\langle \Lambda_{\psi_M}(x), J_M\delta_M^{it}J_M\Lambda_{\widehat{M^{\mathrm{cop}}}}(\lambda_{M^{\mathrm{cop}}}(\omega))\rangle dt \\
&= \langle \Lambda_{\psi_M}(x), \sqrt{\frac{n}{\pi}}\int_{-\infty}^{+\infty} e^{-nt^2}\nu^{-t/2}J_M\delta_M^{it}J_M\Lambda_{\widehat{M^{\mathrm{cop}}}}(\lambda_{M^{\mathrm{cop}}}(\omega))dt\rangle,
\end{aligned}$$

so that $\omega_n \in \mathscr{I}_{M^{\mathrm{cop}}}$ with

$$\Lambda_{\widehat{M^{\mathrm{cop}}}}(\lambda_{M^{\mathrm{cop}}}(\omega_n)) = \sqrt{\frac{n}{\pi}}\int_{-\infty}^{+\infty} e^{-nt^2}\nu^{-t/2}J_M\delta_M^{it}J_M\Lambda_{\widehat{M^{\mathrm{cop}}}}(\lambda_{M^{\mathrm{cop}}}(\omega)).$$

Now a standard calculation shows that the right hand side converges in norm to $\Lambda_{\widehat{M^{\mathrm{cop}}}}(\lambda_{M^{\mathrm{cop}}}(\omega))$ when n goes to infinity. Since we know already that $\lambda_{M^{\mathrm{cop}}}(M_*) \cap \mathscr{N}_{\varphi_{\widehat{M}'}}$ is a σ-strong-norm core for $\Lambda_{\widehat{M^{\mathrm{cop}}}}$, we can conclude from the foregoing calculations that $\lambda_{M^{\mathrm{cop}}}(M_*^\delta) \cap \mathscr{N}_{\varphi_{\widehat{M}'}}$ is a σ-strong-norm core for $\widehat{\Gamma}_M$.

□

We end by quickly recalling the two main classical examples of von Neumann algebraic quantum groups. Let $\mathfrak{G}$ be a locally compact group, with left Haar measure ϱ. Then $M = \mathscr{L}^\infty(\mathfrak{G}, \varrho)$ is a von Neumann algebra, and it becomes a von Neumann algebraic quantum group by defining $\Delta_M(f)$, where f is (the equivalence class of) an essentially bounded function f on $\mathfrak{G}$, to be (the equivalence class of) the function in

$$\mathscr{L}^\infty(\mathfrak{G} \times \mathfrak{G}, \varrho \times \varrho) \cong M \otimes M$$

which assigns $f(gh)$ to $(g, h) \in \mathfrak{G} \times \mathfrak{G}$. Then the invariant weights become integration with respect to the left and right Haar measure, the antipode is dual to the inversion in the group (and in particular, the scaling group is trivial), and the modular element becomes the modular function. One can show that any von Neumann algebraic quantum group with *commutative* underlying von Neumann algebra is of this form.

The second main example is the dual of the previous construction. We consider again a locally compact group $\mathfrak{G}$, and consider its left regular representation π on $\mathscr{L}^2(\mathfrak{G}, \varrho)$, that is, $(\pi(g)f)(h) = f(g^{-1}h)$ for $f \in \mathscr{L}^2(\mathfrak{G}, \varrho)$. Then denote

$$\widehat{M} := \mathscr{L}(\mathfrak{G}) = \{\pi(g) \mid g \in \mathfrak{G}\}''.$$

The application $\Delta_{\widehat{M}}(\pi(g)) = \pi(g)\otimes\pi(g)$ can then be extended (uniquely) to a comultiplication on $\widehat{M}$, making it into a von Neumann algebraic quantum group. The left invariant weight will equal the right invariant weight in this case, and this common weight is then called the *Plancherel weight*. It will be tracial iff the modular function of the group is trivial.

6.2 C*-algebraic quantum groups

Associated to any von Neumann algebraic quantum group, there are two canonical C*-algebraic quantum groups: a reduced one and a universal one, which are resp. smallest and largest among all possible C*-algebraic realizations of the von Neumann algebraic quantum group.[3] Since we will not very often work with C*-algebraic quantum groups directly, we will not recall their definition in detail, only commenting on the structures we will use.

For the following result, we refer to [57].

Definition-Proposition 6.2.1. *Let M be a von Neumann algebraic quantum group. The associated* reduced C*-algebraic quantum group *consists of the norm-closure A of the set $\{(\iota_M \otimes \omega)(W_M) \mid \omega \in \widehat{M}_*\}$, which can be shown to be a C*-algebra, together with the restriction of the map Δ_M to A, which can be shown to have range in $M(A \underset{min}{\otimes} A)$.*

For example, if $\mathfrak{G}$ is a locally compact group, then the reduced C*-algebra of $M = \mathscr{L}^\infty(\mathfrak{G})$ is $A = C_0(\mathfrak{G})$, while the reduced C*-algebra of $\widehat{M} = \mathscr{L}(\mathfrak{G})$ equals $C_r^*(\mathfrak{G})$, the reduced C*-algebra of $\mathfrak{G}$.

The following discussion is taken from [54].

[3]The term 'locally compact quantum group' should then refer to the 'common object' underlying all C*-algebraic implementations of some von Neumann algebraic quantum group.

Definition-Proposition 6.2.2. *Let M be a von Neumann algebraic quantum group. The space $\mathscr{L}^1_*(M)$, consisting of those $\omega \in M_*$ for which the functional $x \to \overline{\omega(S_M(x)^*)}$ on the *-algebra of analytic elements for τ_t^M has a (necessarily unique) extension to a normal functional ω^* on M, is called* the restricted predual of M.

It has a Banach *-algebra structure, by putting

$$\omega_1 \cdot \omega_2 := (\omega_1 \otimes \omega_2) \circ \Delta_M,$$

giving it the *-operation introduced above, and giving it the norm

$$\|\omega\|_{\mathscr{L}^1_*(M)} = max\{\|\omega\|, \|\omega^*\|\}.$$

Definition 6.2.3. *Let M be a von Neumann algebraic quantum group. The* universal C*-algebra A^u *associated to M is the universal C*-algebraic envelope of the Banach *-algebra $\mathscr{L}^1_*(\widehat{M})$.*

Similarly, there is a universal C*-algebra associated with the dual von Neumann algebraic quantum group $\widehat{M}$, and we denote it by the symbol $\widehat{A}^u$. One can show that A^u also has the structure of a C*-algebraic quantum group, but we will not be concerned with it in this thesis. The main use of the universal C*-algebra is that its non-degenerate *-representations are in one-to-one correspondence with the unitary corepresentations of the dual von Neumann algebraic quantum group.

Proposition 6.2.4. *Let M be a von Neumann algebraic quantum group. Then any unitary left corepresentation U is* continuous*:*

$$U \in M(A \underset{min}{\otimes} B_0(\mathscr{H})).$$

It then gives rise to a non-degenerate *-representation of $\widehat{A}^u$ by extending

$$\mathscr{L}^1_*(M) \to B(\mathscr{H}) : \omega \to (\omega \otimes \iota_{B(\mathscr{H})})(U),$$

which can be shown to be multiplicative and *-preserving, to $\widehat{A}^u$.

Moreover, there exists a universal unitary left corepresentation

$$W^u \in M(A \underset{min}{\otimes} \widehat{A}^u) \subseteq M \otimes B(\mathscr{H}^u)$$

on a Hilbert space $\mathscr{H}^u$, such that any *non-degenerate *-representation π of $\widehat{A}^u$ is of the above form, with associated unitary corepresentation*

$$U = (\iota_A \otimes \pi)(W^u).$$

Unitary *right* corepresentations then correspond one-to-one to non-degenerate right *-representations of $\widehat{A}^u$.

When $\mathfrak{G}$ is a locally compact group, it is easy to show that for $\widehat{M} = \mathscr{L}(\mathfrak{G})$, the associated universal C*-algebra equals the universal C*-algebra of $\mathfrak{G}$, and then the above result says that there is a natural one-to-one correspondence between unitary corepresentations of $\mathscr{L}^\infty(\mathfrak{G})$ and unitary representations of $\mathfrak{G}$.

6.3 Coactions of von Neumann algebraic quantum groups

We recall in this section some definitions and results from [85]. We warn however that that paper works in the setting of *left* coactions, while we will mostly work with *right* coactions, so we will left-right translate the notions of [85]. One can do this easily by replacing a von Neumann algebraic quantum group M by its coopposite M^{cop}.

Definition 6.3.1. *Let N be a von Neumann algebra, M a von Neumann algebraic quantum group, and $\alpha : N \to N \bar{\otimes} M$ a normal unital *-homomorphism. We call α a* right coaction of M on N *if α is injective and*

$$(\alpha \otimes \iota_M)\alpha = (\iota_N \otimes \Delta_M)\alpha.$$

We call α faithful *when the algebra generated by the set $\{(\omega \otimes \iota_M)\alpha(x) \mid x \in N, \omega \in N_*\}$ is σ-weakly dense in M.*

We call α integrable *when $\mathscr{M}_{\iota_N \otimes \varphi_M} \cap \alpha(N)$ is σ-weakly dense in $\alpha(N)$.*

We call the von Neumann algebra

$$N^\alpha := \{x \in N \mid \alpha(x) = x \otimes 1_M\}$$

the algebra of coinvariants of α, *and we say that α is* ergodic *when $N^\alpha = \mathbb{C} \cdot 1_N$.*

In case $M = \mathscr{L}^\infty(\mathfrak{G})$ for a locally compact group $\mathfrak{G}$, a coaction of M on a von Neumann algebra N is the same as a continuous homomorphism $\mathfrak{G} \to \text{Aut}(N)$, where $\text{Aut}(N)$ denotes the group of *-automorphisms of N, endowed with the point-σ-weak topology. The faithfulness of the coaction then corresponds to $\alpha_g \neq \iota_N$ for g not the unit element of the group, while the ergodicity corresponds to having $x \in N$ scalar when $\alpha_g(x) = x$ for all $g \in \mathfrak{G}$.

We introduce some further terminology concerning coactions.

Definition 6.3.2. *Let N be a von Neumann algebra, M and P two von Neumann algebraic quantum groups, and $\alpha : N \to N \otimes M$ a right coaction of M on N, $\gamma : N \to P \otimes N$ a left coaction of P on N. Then we say that α and γ* commute *if*

$$(\gamma \otimes \iota_M)\alpha = (\iota_P \otimes \alpha)\gamma.$$

Definition 6.3.3. *Let N be a von Neumann algebra, and α a right coaction of a von Neumann algebraic quantum group M on N. Then an nsf weight ψ on N is called* invariant w.r.t. α *if for any $\omega \in M_*^+$ and $x \in \mathscr{M}_\psi^+$, we have*

$$\psi((\iota \otimes \omega)\alpha(x)) = \omega(1)\psi(x).$$

More generally, if m is a positive operator affiliated with M, we say that an nsf weight ψ is m-invariant w.r.t. α *when for all $\xi \in \mathscr{D}(m^{1/2})$ and $x \in \mathscr{M}_\psi^+$, we have*

$$\psi((\iota \otimes \omega_{\xi,\xi})\alpha(x)) = \psi(x)\|m^{1/2}\xi\|^2.$$

Definition 6.3.4. *Let M be a von Neumann algebraic quantum group, N a von Neumann algebra, and α a right coaction of M on N. A* 1-cocycle for the coaction α *(also called* α-cocycle*) is a unitary element $v \in N \otimes M$ which satisfies*

$$(\iota_N \otimes \Delta_M)(v) = v_{12}(\alpha \otimes \iota_M)(v).$$

If α_1 and α_2 are two right coactions of M on N, then α_1 and α_2 are called cocycle equivalent *or* outer equivalent *if there exists an α_1-cocycle v such that $\alpha_2(x) = v\alpha_1(x)v^*$ for $x \in N$.*

These notions then agree with those introduced in Definition 5.2.7 in case $M = \mathscr{L}^\infty(\mathbb{R})$.
We now give some information concerning the further structure associated to a general coaction.

First of all, we can characterize the image of any coaction α as follows (Theorem 2.7 of [85], which refers to [32]).

Proposition 6.3.5. *Let M be a von Neumann algebraic quantum group, and α a right coaction of M on N. Then*

$$\alpha(N) = \{z \in N \otimes M \mid (\alpha \otimes \iota_M)(z) = (\iota_N \otimes \Delta_M)(z)\}.$$

Next, we have that from any coaction, we can construct a new von Neumann algebra.

Definition 6.3.6. *Let α be a right coaction of a von Neumann algebraic quantum group M on a von Neumann algebra N. Then the* crossed product von Neumann algebra $N \underset{\alpha}{\rtimes} M$ *(denoted $N \rtimes M$ when α is clear) is the σ-weak closure of the linear span of*

$$\{(1 \otimes m)\alpha(x) \mid x \in N, m \in \widehat{M}'\} \subseteq B(\mathscr{L}^2(N) \otimes \mathscr{L}^2(M)).$$

It is not so difficult to show that this is indeed a von Neumann algebra (i.e., closed under multiplication and the *-operation).

Definition-Proposition 6.3.7. *Let α be a right coaction of a von Neumann algebraic quantum group M on a von Neumann algebra N. Then the assignment*

$$(1 \otimes m)\alpha(x) \rightarrow (1 \otimes \Delta_{\widehat{M}'}(m))(\alpha(x) \otimes 1)$$

extends to a well-defined integrable *coaction*

$$\hat{\alpha} : N \rtimes M \rightarrow (N \rtimes M) \otimes \widehat{M}',$$

called the dual coaction of α. *The algebra of coinvariants $(N \rtimes M)^{\hat{\alpha}}$ equals $\alpha(N)$.*

The next definition describes the *dual weight* construction, which allows one to canonically lift nsf weights on N to nsf weights on $N \rtimes M$. We first recall a result from [85] (Prop. 1.3): if α is a right coaction of a von Neumann algebraic quantum group M on a von Neumann algebra N, then the assignment

$$x \in N^+ \rightarrow N^{+,\mathrm{ext}} : x \rightarrow (\iota \otimes \varphi_M)\alpha(x)$$

can be interpreted as a faithful normal N^α-valued weight T_α on N. In particular, a coaction α is integrable iff this operator valued weight T_α is semi-finite.

Definition-Proposition 6.3.8. *Let α be a right coaction of a von Neumann algebraic quantum group M on a von Neumann algebra N. Let φ_N be an nsf weight on N. Let $T_{\widehat{\alpha}} : (N \rtimes M)^+ \to (\alpha(N))^{+,ext}$ be the nsf operator valued weight $(\iota_{N\rtimes M} \otimes \varphi_{\widehat{M}'})\widehat{\alpha}$. Then the weight $\varphi_{N\rtimes M} := \varphi_N \circ \alpha^{-1} \circ T_{\widehat{\alpha}}$ on $N \rtimes M$ is called* the dual weight of φ_N *(w.r.t. α). There is a natural semi-cyclic representation for $\varphi_{N\rtimes M}$ on $\mathscr{L}^2(N) \otimes \mathscr{L}^2(M)$, by closing the map*

$$\sum_i (1 \otimes m_i)\alpha(x_i) \to \sum_i \Lambda_{\varphi_N}(x_i) \otimes \Lambda_{\widehat{M}'}(m_i),$$

defined on the linear span of the set $\{(1 \otimes m)\alpha(x) \mid m \in \mathscr{N}_{\varphi_{\widehat{M}'}}, x \in \mathscr{N}_{\varphi_N}\}$, in the ($\sigma$-strong)-(norm)-topology.*

It is shown in [85] that the resulting identification of $\mathscr{L}^2(N \rtimes M)$ and $\mathscr{L}^2(N) \otimes \mathscr{L}^2(M)$ is in fact independent of the choice of nsf weight on N. In the following, we will then always transport the structure from $\mathscr{L}^2(N \rtimes M)$ to $\mathscr{L}^2(N) \otimes \mathscr{L}^2(M)$ via this correspondence.

We have the following relation between the modular one-parametergroups of a weight φ_N and its dual (cf. Proposition 5.6.3).

Proposition 6.3.9. *Let α be a right coaction of a von Neumann algebraic quantum group M on a von Neumann algebra N. Let φ_N be an nsf weight on N, and $\varphi_{N\rtimes M}$ the dual nsf weight on $N \rtimes M$. Then*

$$\alpha \circ \sigma_t^{\varphi_N} = \sigma_t^{\varphi_{N\rtimes M}} \circ \alpha.$$

In our next chapters, we will be mainly concerned with *integrable* coactions. The following easy lemma concerning integrable coactions is used to recall an important Cauchy-Schwarz type inequality.

Lemma 6.3.10. *Let α be an integrable right coaction of a von Neumann algebraic quantum group M on a von Neumann algebra N. Then if $x \in \mathscr{N}_{T_\alpha}$, we have $(\omega \otimes \iota_M)\alpha(x) \in \mathscr{N}_{\varphi_M}$ for all $\omega \in N_*$.*

Proof. This follows from the inequality

$$((\omega \otimes \iota_M)\alpha(x))^*((\omega \otimes \iota_M)\alpha(x)) \leqslant \|\omega\| \cdot (|\omega| \otimes \iota_M)(\alpha(x^*x)),$$

where $|\omega|$ is the *absolute value* of ω. □

Our next definition-proposition again recalls a result of [85], namely the construction, for an arbitrary right coaction α, of a certain unitary in $B(\mathscr{L}^2(N)) \otimes M$ *implementing* the coaction.

Definition-Proposition 6.3.11. *Let $\alpha : N \to N \otimes M$ be a right coaction of a von Neumann algebraic quantum group M on a von Neumann algebra N. Then $U := J_{N \rtimes M}(J_N \otimes J_{\widehat{M}})$ is a unitary right corepresentation, implementing α in the following way:*

$$U(x \otimes 1)U^* = \alpha(x) \qquad \text{for all } x \in N.$$

It is called the (canonical) unitary implementation *of α.*

When α is integrable, one has the following alternative formula for U. Let μ be an nsf weight on N^α, and let φ_N be the nsf weight $\mu \circ T_\alpha$ on N. Then for any $\xi, \eta \in \mathscr{L}^2(M)$ with $\xi \in \mathscr{D}(\delta_M^{-1/2})$, and any $x \in \mathscr{N}_{\varphi_N}$, we have

$$(\iota \otimes \omega_{\xi,\eta})(U)\Lambda_{\varphi_N}(x) = \Lambda_{\varphi_N}((\iota \otimes \omega_{\delta_M^{-1/2}\xi,\eta})(\alpha(x))),$$

where the right hand side can be shown to be well-defined.

We remark that by the closedness of Λ_{φ_N}, it is easily seen that the alternative formula for U stays true if we replace $\omega_{\xi,\eta}$ on the left side by a general $\omega \in M_*$ for which the function $x \to \omega(x\delta_M^{-1/2})$ extends from the linear space of left multipliers of $\delta_M^{-1/2}$ to a normal functional ω_δ on M, and $\omega_{\delta_M^{-1/2}\xi,\eta}$ on the right side by this ω_δ.

In what follows, it will at times be more convenient to work with $\mathscr{L}^2(M) \otimes \mathscr{L}^2(N)$ in stead of $\mathscr{L}^2(N) \otimes \mathscr{L}^2(M)$. We then consider also *this* space as a natural $N \rtimes M$-equivalence correspondence, using the flip map to transport all structure from $\mathscr{L}^2(N) \otimes \mathscr{L}^2(M)$.

The following theorem of [85] will be of the most importance to us.

Theorem 6.3.12. *Let N be a von Neumann algebra, M a von Neumann algebraic quantum group, and α a right coaction of M on N. Let U be the unitary implementation of α. Then α is integrable iff the map*

$$\{(1 \otimes (\iota \otimes \omega)(V_M))\alpha(x) \mid x \in N, \omega \in M_*\} \to B(\mathscr{L}^2(N)) :$$

$$(1 \otimes (\iota \otimes \omega)(V_M))\alpha(x) \to (\iota \otimes \omega)(U) \cdot x$$

extends to a normal **-homomorphism*

$$\rho_\alpha : N \rtimes M \to B(\mathscr{L}^2(N)).$$

It is not difficult to show that the range of such a map ρ_α is then precisely N_2, with N_2 the basic construction applied to $N^\alpha \subseteq N$.

Definition 6.3.13. *Let α be an integrable right coaction of a von Neumann algebraic quantum group M on a von Neumann algebra N. We call the map $\rho_\alpha : N \rtimes M \to N_2$ the* Galois homomorphism *associated to α.*

By the map ρ_α, we can make normal unital left and right $N \rtimes M$-*-representations on $\mathscr{L}^2(N)$. We have then also associated left and right $\widehat{M}'$-*-representations. We denote them respectively by $\hat{\pi}'_\alpha$ (so $\hat{\pi}'_\alpha(m) = \rho_\alpha(1 \otimes m)$ for $m \in \widehat{M}'$) and $\hat{\theta}'_\alpha$ (so $\hat{\theta}'_\alpha(m) = J_N \hat{\pi}'_\alpha(m)^* J_N$ when $m \in \widehat{M}'$). Finally, by $\hat{\pi}_\alpha$ and $\hat{\theta}_\alpha$ we denote the associated left and right $\widehat{M}$-module structures (so $\hat{\pi}_\alpha = \hat{\theta}'_\alpha \circ C_{\widehat{M}}$ and $\hat{\theta}_\alpha = \hat{\pi}_\alpha \circ C_{\widehat{M}}$).

Lemma 6.3.14. *If α is an integrable right coaction of a von Neumann algebraic quantum group M on a von Neumann algebra N, then $\hat{\theta}'_\alpha(m) = \hat{\pi}'_\alpha(R_{\widehat{M}'}(m))$.*

Proof. Just use that $(J_N \otimes J_{\widehat{M}})U(J_N \otimes J_{\widehat{M}}) = U^*$ and $(J_M \otimes J_{\widehat{M}})V_M(J_M \otimes J_{\widehat{M}}) = V_M^*$.

□

In case of an integrable coaction, also the modular *operators* of an nsf weight $\varphi_N = \mu \circ T_\alpha$ and its dual weight $\varphi_{N \rtimes M}$ can be related.

Proposition 6.3.15. *Let α be an integrable right coaction of a von Neumann algebraic quantum group M on a von Neumann algebra N. Let μ be an nsf weight on N^α, let φ_N be the nsf weight $\mu \circ T_\alpha$ on N, and $\varphi_{N \rtimes M}$ the dual weight of φ_N on $N \rtimes M$. Denote by κ_t^M the one-parametergroup of automorphisms on M, given by*

$$\kappa_t^M(x) = \delta_M^{-it} \tau_{-t}^M(x) \delta_M^{it}.$$

Then φ_M is κ_t^M-invariant, and if we denote by q_M^{it} the resulting one-parametergroup of unitaries determined by

$$q_M^{it} \Lambda_{\varphi_M}(x) = \Lambda_{\varphi_M}(\kappa_t^M(x)), \qquad x \in \mathscr{N}_{\varphi_N},$$

then

$$\nabla^{it}_{\varphi_{N\rtimes M}} = \nabla^{it}_{\varphi_N} \otimes q^{it}_M.$$

Proof. The proof of this result is contained in the proof of Proposition 4.3 of [85]. □

The one-parametergroup q^{it}_M appearing in the previous proof also has a different expression:

$$q^{it}_M = \delta^{-it}_M \nabla^{-it}_{\widehat{M}}.$$

We introduce some further notations for an integrable coaction α of a von Neumann algebraic quantum group M on N. By $\widehat{N}$, we will mean the space of right $\widehat{M}$-intertwiners between $\mathscr{L}^2(M)$ and $\mathscr{L}^2(N)$. We then also denote by $\hat{O}$ the space of right $\widehat{M}$-intertwiners between $\mathscr{L}^2(N)$ and $\mathscr{L}^2(M)$, and by $\widehat{P}$ the space of right $\widehat{M}$-intertwiners from $\mathscr{L}^2(N)$ to itself (so $\widehat{P} = \hat{\theta}_\alpha(\widehat{M})'$). We further denote $\widehat{Q} = \begin{pmatrix} \widehat{P} & \widehat{N} \\ \hat{O} & \widehat{M} \end{pmatrix}$. Note that when ρ_α is *faithful*, $\mathscr{L}^2(N)$ is a $\widehat{P}$-$\widehat{M}$-equivalence correspondence, and $\widehat{Q}$ a linking algebra between $\widehat{M}$ and $\widehat{P}$.

6.4 More on integrable coactions

In this section, we give some further results concerning integrable coactions. Apart from proving some commutation relations, which will be of importance in the following chapter, our main result is Theorem 6.4.8, which gives an alternative description, on the Hilbert space level, of the Galois homomorphism of an integrable coaction.

Throughout, M will denote a von Neumann algebraic quantum group, N a von Neumann algebra, and α an integrable right coaction of M on N, whose unitary implementation we denote by U. We also suppose that N^α comes equipped with some fixed nsf weight μ, and then we denote $\varphi_N = \mu \circ T_\alpha$. Recall that $\mathscr{T}_{\varphi_N,T_\alpha}$ denotes the Toimta algebra for φ_N and the operator valued weight $T_\alpha = (\iota \otimes \varphi)\alpha$ (cf. page 175).

Lemma 6.4.1. *The map*

$$\Lambda_{\varphi_N}(\mathscr{T}_{\varphi_N,T_\alpha}) \underset{\mu}{\otimes} \Lambda_{\varphi_N}(\mathscr{T}_{\varphi_N,T_\alpha}) \to \mathscr{L}^2(N) \otimes \mathscr{L}^2(M):$$

$$\Lambda_{\varphi_N}(x) \underset{\mu}{\otimes} \Lambda_{\varphi_N}(y) \to (\Lambda_{\varphi_N} \otimes \Lambda_{\varphi_M})(\alpha(x)(y \otimes 1))$$

is well-defined and isometric.

Proof. Using the formula of Lemma 5.7.4 to evaluate the scalar product of left hand side elements, this is easy. □

Denote by
$$G : \mathscr{L}^2(N) \underset{\mu}{\otimes} \mathscr{L}^2(N) \to \mathscr{L}(N) \otimes \mathscr{L}^2(M)$$
the closure of the previous map.[4]

Definition 6.4.2. *Let α be an integrable right coaction of a von Neumann algebraic quantum group M on a von Neumann algebra N. We call the operator*
$$\tilde{G} = \Sigma G : \mathscr{L}^2(N) \underset{\mu}{\otimes} \mathscr{L}^2(N) \to \mathscr{L}^2(M) \otimes \mathscr{L}^2(N)$$
the Galois map *or the* Galois isometry *for α.*

Remark: The reason for putting a flip map in front of G, is to make it right N-linear in such a way that this is just right N-linearity on the second factors of the domain and range, so that 'the second leg of $\tilde{G}$ is in N'. See the third commutation relation in Lemma 6.4.10.

Our aim is to prove that the Galois isometry for an integrable coaction is a unitary iff the Galois homomorphism for the action is faithful (Theorem 6.4.8). We need some preparation for this.

Let $N^\alpha \underset{T_\alpha}{\subseteq} N \underset{T_2}{\subseteq} N_2$ be the basic construction for T_α (see Definition 5.7), and denote $\varphi_2 = \varphi_N \circ T_2$. Recall that we identify $\mathscr{L}^2(N) \underset{\mu}{\otimes} \mathscr{L}^2(N)$ and $\mathscr{L}^2(N_2)$ (cf. Theorem 5.7.5).

Lemma 6.4.3. *If $m \in \mathscr{N}_{\varphi_{\widehat{M}'}}$ and $z \in \mathscr{N}_{\varphi_N}$, then $\rho_\alpha((1 \otimes m)\alpha(z)) \in \mathscr{N}_{\varphi_2}$ and*
$$G^*(\Lambda_{\varphi_N}(z) \otimes \widehat{\Gamma}_M(m)) = \Lambda_{\varphi_2}(\rho_\alpha((1 \otimes m)\alpha(z))).$$

Proof. Choose $m \in \mathscr{N}_{\varphi_{\widehat{M}'}}$ of the form $(\iota \otimes \omega)(V_M^*)$, with $\omega \in M_*$ such that $x \to \overline{\omega}(x\delta_M^{-1/2})$ coincides with a normal functional $\overline{\omega}_\delta$ on the set of left

[4] This corresponds to the map denoted as H in section 2.5 of the first part of this thesis.

multipliers of $\delta_M^{-1/2}$ in M. Then we have, for $x, y \in \mathscr{T}_{\varphi_N, T_\alpha}$ and $z \in \mathscr{N}_{\varphi_N}$, that

$$\begin{aligned}
&\langle \Lambda_{\varphi_N}(x), \widehat{\theta}'_\alpha(m) z \Lambda_{\varphi_N}(\sigma_{-i}^{\varphi_N}(y^*))\rangle \\
&\quad = \langle \Lambda_{\varphi_N}(x), (\iota \otimes \omega)(U^*) z \Lambda_{\varphi_N}(\sigma_{-i}^{\varphi_N}(y^*))\rangle \\
&\quad = \langle (\iota \otimes \overline{\omega})(U) \Lambda_{\varphi_N}(x), z \Lambda_{\varphi_N}(\sigma_{-i}^{\varphi_N}(y^*))\rangle \\
&\quad = \langle \Lambda_{\varphi_N}((\iota \otimes \overline{\omega}_\delta)(\alpha(x))), \Lambda_{\varphi_N}(z \sigma_{-i}^{\varphi_N}(y^*))\rangle \\
&\quad = \varphi_N(\sigma_i^{\varphi_N}(y) z^* (\iota \otimes \overline{\omega}_\delta)(\alpha(x))) \\
&\quad = \varphi_N(z^* (\iota \otimes \overline{\omega}_\delta)(\alpha(x)) y) \\
&\quad = \overline{\omega}_\delta((\omega_{\Lambda_{\varphi_N}(y), \Lambda_{\varphi_N}(z)} \otimes \iota)\alpha(x)),
\end{aligned}$$

using the KMS property. But since for $a \in \mathscr{N}_{\varphi_M}$, we have

$$\langle \Lambda_{\varphi_M}(a), \widehat{\Gamma}_M(m)\rangle = \overline{\omega}_\delta(a),$$

by Lemma 6.1.20, this last expression equals

$$\langle G(\Lambda_{\varphi_N}(x) \underset{\mu}{\otimes} \Lambda_{\varphi_N}(y)), \Lambda_{\varphi_N}(z) \otimes \widehat{\Gamma}_M(m)\rangle.$$

Since such m form a σ-strong-norm core for $\widehat{\Gamma}_M$, again by Lemma 6.1.20, we have

$$\langle \Lambda_{\varphi_N}(x), \widehat{\theta}'_\alpha(m) z \Lambda_{\varphi_N}(\sigma_{-i}^{N}(y^*))\rangle = \langle G(\Lambda_{\varphi_N}(x) \underset{\mu}{\otimes} \Lambda_{\varphi_N}(y)), \Lambda_{\varphi_N}(z) \otimes \widehat{\Gamma}_M(m)\rangle,$$

for all $m \in \mathscr{N}_{\varphi_{\widehat{M}'}}$. By the second part of Lemma 5.7.7, we then get

$$\rho_\alpha((1 \otimes m)\alpha(z)) \in \mathscr{N}_{\varphi_2}$$

and

$$\Lambda_{\varphi_2}(\rho_\alpha((1 \otimes m)\alpha(z))) = G^*(\Lambda_{\varphi_N}(z) \otimes \widehat{\Gamma}_M(m))$$

for all $m \in \mathscr{N}_{\varphi_{\widehat{M}'}}$ and $z \in \mathscr{N}_{\varphi_N}$.

□

Lemma 6.4.4. *The isometry G is a left $N \rtimes M$-module morphism.*

Proof. For $x, y, z \in \mathscr{T}_{\varphi_N, T_\alpha}$, we have

$$\begin{aligned}
G\pi_{N_2}(x)(\Lambda_{\varphi_N}(y) \underset{\mu}{\otimes} \Lambda_{\varphi_N}(z)) &= G(\Lambda_{\varphi_N}(xy) \underset{\mu}{\otimes} \Lambda_{\varphi_N}(z)) \\
&= (\Lambda_{\varphi_N} \otimes \Lambda_{\varphi_M})(\alpha(xy)(z \otimes 1)) \\
&= \alpha(x) G(\Lambda_{\varphi_N}(y) \underset{\mu}{\otimes} \Lambda_{\varphi_N}(z)).
\end{aligned}$$

Hence $G\pi_{N_2}(x) = \alpha(x)G$ for all $x \in \mathscr{T}_{\varphi_N,T_\alpha}$, and then this is also true for all $x \in N$. Further, if $m \in \widehat{M}'$, $n \in \mathscr{N}_{\varphi_{\widehat{M}'}}$ and $z \in \mathscr{N}_{\varphi_N}$, then

$$\rho_\alpha((1 \otimes mn)\alpha(z)) \in \mathscr{N}_{\varphi_2}$$

by the previous lemma, and we have

$$\begin{aligned}\rho_\alpha(1 \otimes m)G^*(\Lambda_{\varphi_N}(z) \otimes \widehat{\Gamma}_M(n)) &= \Lambda_{\varphi_2}(\rho_\alpha((1 \otimes mn)\alpha(z))) \\ &= G^*(\Lambda_{\varphi_N}(z) \otimes \widehat{\Gamma}_M(mn)),\end{aligned}$$

hence $G\,\rho_\alpha(1 \otimes m) = (1 \otimes m)G$ for all $m \in \widehat{M}'$. Since $N \rtimes M$ is generated by $1 \otimes \widehat{M}'$ and $\alpha(N)$, the lemma is proven. □

Remark: This implies that $\pi_{N_2}(\rho_\alpha(x)) = G^*xG$ for $x \in N \rtimes M$, as G is an isometry.

Lemma 6.4.5. *The following commutation relations hold:*

1. $\nabla^{it}_{\varphi_{N \rtimes M}} G = G \nabla^{it}_{\varphi_2}$,
2. $J_{N \rtimes M} G = G J_{N_2}$.

Proof. Using Corollary 5.7.6, the first commutation relation reduces to proving that for $x, y \in \mathscr{T}_{\varphi_N,T_\alpha}$, we have

$$\nabla^{it}_{\varphi_{N \rtimes M}}((\Lambda_{\varphi_N} \otimes \Lambda_{\varphi_M})(\alpha(x)(y \otimes 1))) = (\Lambda_{\varphi_N} \otimes \Lambda_{\varphi_M})(\alpha(\sigma^{\varphi_N}_t(x))(\sigma^{\varphi_N}_t(y) \otimes 1)).$$

Combining Propositions 6.3.9 and 6.3.15, and using their notation, we have that for $x, y \in \mathscr{T}_{\varphi_N,T_\alpha}$ and $\xi \in \mathscr{L}^2(M)$, the following identity holds:

$$\nabla^{it}_{\varphi_{N \rtimes M}}(\alpha(x)(\Lambda_{\varphi_N}(y) \otimes \xi)) = \alpha(\sigma^{\varphi_N}_t(x))(\Lambda_{\varphi_N}(\sigma^{\varphi_N}_t(y)) \otimes q^{it}_M \xi).$$

Now let $a \in \mathscr{T}_{\varphi_M}$. Since σ^M_t commutes with κ^M_t, still using the notation of Proposition 6.3.15, we have that $\kappa^M_t(a)$ is then also in $\mathscr{T}_{\varphi_M}$, with

$$\sigma^{\varphi_M}_z(\kappa^M_t(a)) = \kappa^M_t(\sigma^{\varphi_M}_z(a)) \qquad \text{for } t \in \mathbb{R}, z \in \mathbb{C}.$$

Hence for $a \in \mathscr{T}_{\varphi_M}$, and $x, y \in \mathscr{T}_{\varphi_N,T_\alpha}$, we get

$$\begin{aligned}&\nabla^{it}_{\varphi_{N \rtimes M}}(1 \otimes J_M \sigma^{\varphi_M}_{i/2}(a)^* J_M)((\Lambda_{\varphi_N} \otimes \Lambda_{\varphi_M})(\alpha(x)(y \otimes 1))) \\ &= \nabla^{it}_{\varphi_{N \rtimes M}}(\Lambda_{\varphi_N} \otimes \Lambda_{\varphi_M})(\alpha(x)(y \otimes a)) \\ &= (\Lambda_{\varphi_N} \otimes \Lambda_{\varphi_M})(\alpha(\sigma^{\varphi_N}_t(x))(\sigma^{\varphi_N}_t(y) \otimes \kappa^M_t(a))) \\ &= (1 \otimes J_M \kappa^M_t(\sigma^{\varphi_M}_{i/2}(a))^* J_M)(\Lambda_{\varphi_N} \otimes \Lambda_{\varphi_M})(\alpha(\sigma^{\varphi_N}_t(x))(\sigma^{\varphi_N}_t(y) \otimes 1)),\end{aligned}$$

and letting $\sigma^{\varphi_M}_{i/2}(a)$ tend to 1_M σ-strongly, we see that

$$\nabla^{it}_{\varphi_{N\rtimes M}}((\Lambda_{\varphi_N}\otimes\Lambda_{\varphi_M})(\alpha(x)(y\otimes 1))) = (\Lambda_{\varphi_N}\otimes\Lambda_{\varphi_M})(\alpha(\sigma^{\varphi_N}_t(x))(\sigma^{\varphi_N}_t(y)\otimes 1)),$$

which proves the first commutation relation.

From this, it follows that $G^*\nabla^{1/2}_{\varphi_{N\rtimes M}}$ will equal the restriction of $\nabla^{1/2}_{\varphi_2}G^*$ to $\mathscr{D}(\nabla^{1/2}_{\varphi_{N\rtimes M}})$. Denote $\mathcal{T}_{\varphi_{N\rtimes M}} = J_{N\rtimes M}\nabla^{1/2}_{\varphi_{N\rtimes M}}$ and $\mathcal{T}_{\varphi_2} = J_{N_2}\nabla^{1/2}_{\varphi_2}$, where we recall that $\mathcal{T}$ denotes the Hilbert space implementation (w.r.t. the given weight) of the *-operation on the von Neumann algebra. Then $\mathcal{T}_{\varphi_2}G^* = J_{N_2}G^*\nabla^{1/2}_{\varphi_{N\rtimes M}}$ on $\mathscr{D}(\nabla^{1/2}_{\varphi_{N\rtimes M}})$. So to prove the second commutation relation (in the form $G^*J_{N\rtimes M} = J_{N_2}G^*$), we only have to find a subset K of $\mathscr{D}(\nabla^{1/2}_{\varphi_{N\rtimes M}}) = \mathscr{D}(\mathcal{T}_{\varphi_{N\rtimes M}})$ whose image under $\nabla^{1/2}_{\varphi_{N\rtimes M}}$ (or $\mathcal{T}_{\varphi_{N\rtimes M}}$) is dense in $\mathscr{L}^2(N\rtimes M)$, and on which $\mathcal{T}_{\varphi_2}G^*$ and $G^*\mathcal{T}_{\varphi_{N\rtimes M}}$ agree. But take

$$K = \operatorname{span}\{\alpha(x)\Lambda_{\varphi_{N\rtimes M}}((1\otimes m)\alpha(y)) \mid x, y \in \mathscr{T}_{\varphi_N,T_\alpha}, m \in \mathscr{N}_{\varphi_{\widehat{M}'}} \cap \mathscr{N}^*_{\varphi_{\widehat{M}'}}\}.$$

Then clearly $K \subseteq \mathscr{D}(\mathcal{T}_{\varphi_{N\rtimes M}})$ and $\mathcal{T}_{\varphi_{N\rtimes M}}(K) = K$, since

$$\mathcal{T}_{\varphi_{N\rtimes M}}(\alpha(x)\Lambda_{\varphi_{N\rtimes M}}((1\otimes m)\alpha(y))) = \alpha(y^*)\Lambda_{\varphi_{N\rtimes M}}((1\otimes m^*)\alpha(x^*)).$$

Furthermore, if $x, y \in \mathscr{T}_{\varphi_N,T_\alpha}$ and $m \in \mathscr{N}_{\varphi_{\widehat{M}'}} \cap \mathscr{N}^*_{\varphi_{\widehat{M}'}}$, we get from Lemma 6.4.3 and Lemma 6.4.4 that

$$\rho_\alpha(\alpha(x)(1\otimes m)\alpha(y)) \text{ and } \rho_\alpha(\alpha(y^*)(1\otimes m^*)\alpha(x^*))$$

are both in $\mathscr{N}_{\varphi_2}$, and that

$$G^*\alpha(x)\Lambda_{\varphi_{N\rtimes M}}((1\otimes m)\alpha(y)) \in \mathscr{D}(\mathcal{T}_{\varphi_2}),$$

with

$$\begin{aligned}
\mathcal{T}_{\varphi_2}G^*\alpha(x)\Lambda_{\varphi_{N\rtimes M}}((1\otimes m)\alpha(y)) &= \mathcal{T}_{\varphi_2}G^*\alpha(x)(\Lambda_{\varphi_N}(y)\otimes\widehat{\Gamma}_M(m)) \\
&= \mathcal{T}_{\varphi_2}\Lambda_{\varphi_2}(\rho_\alpha(\alpha(x)(1\otimes m)\alpha(y))) \\
&= \Lambda_{\varphi_2}(\rho_\alpha(\alpha(y^*)(1\otimes m^*)\alpha(x^*))) \\
&= G^*\alpha(y^*)\Lambda_{\varphi_{N\rtimes M}}((1\otimes m^*)\alpha(x^*)) \\
&= G^*\mathcal{T}_{\varphi_{N\rtimes M}}\alpha(x)\Lambda_{\varphi_{N\rtimes M}}((1\otimes m)\alpha(y)).
\end{aligned}$$

Since K is dense in $\mathscr{L}^2(N\rtimes M)$, the second commutation relation is proven. □

Corollary 6.4.6. *The map G is a right $N\rtimes M$-module map.*

Proof. This follows from the commutation of G with the modular conjugations: for $x \in N_2$, we have

$$\begin{aligned} G(\theta_{N_2}(x)) &= G(J_{N_2}\pi_{N_2}(x^*)J_{N_2}) \\ &= J_{N \rtimes M}\pi_{N \rtimes M}(x^*)J_{N \rtimes M}G \\ &= \theta_{N \rtimes M}(x)G. \end{aligned}$$

□

Denote by p the central projection in $N \rtimes M$ such that

$$\ker(\rho_\alpha) = (1-p)(N \rtimes M).$$

Lemma 6.4.7. *The projection GG^* equals p.*

Proof. By Lemma 6.4.4, G is a left $N \rtimes M$-module morphism, hence $GG^* \in (N \rtimes M)'$, and $GG^* \leqslant p$ since $G^*pG = \rho_\alpha(p) = 1$. By the previous lemma, GG^* commutes with $J_{N \rtimes M}$, hence GG^* is in the center $\mathscr{Z}(N \rtimes M)$. Since $\rho_\alpha(GG^*) = G^*(GG^*)G = 1$, we must have $GG^* = p$.

□

Theorem 6.4.8. *Let M be a von Neumann algebraic quantum group, and α an integrable coaction of M on a von Neumann algebra N. Then the Galois homomorphism $\rho_\alpha : N \rtimes M \to N_2$ is faithful iff the Galois isometry $\tilde{G}$ is a unitary.*

Proof. This is an immediate corollary of the previous lemma, since G is unitary iff $p = 1$ iff ρ_α is faithful. □

We show now that φ_{N_2} coincides with a weight introduced in [85]. We keep using the notation introduced just before Lemma 6.4.7. Denote further by $(\rho_\alpha)_p$ the restriction of $\rho_\alpha : N \rtimes M \to N_2$ to $p(N \rtimes M)$, and by $\tilde{\varphi}_2$ the nsf weight $\varphi_{N \rtimes M} \circ (\rho_\alpha)_p^{-1}$ on N_2.

Proposition 6.4.9. *The weight $\tilde{\varphi}_2$ equals φ_2.*

Proof. If $m \in \mathscr{N}_{\varphi_{\widehat{M}'}}$ and $z \in \mathscr{N}_{\varphi_N}$, then $\rho_\alpha((1 \otimes m)\alpha(z)) \in \mathscr{N}_{\tilde{\varphi}_2}$, and we can make a semi-cyclic representation $\tilde{\Lambda}$ for $\tilde{\varphi}_2$ on $p(\mathscr{L}^2(N) \otimes \mathscr{L}^2(M))$, determined by

$$\begin{aligned} \tilde{\Lambda}(\rho_\alpha((1 \otimes m)\alpha(z))) &:= \nu_M^{i/8}\, p(\Lambda_{\varphi_{N \rtimes M}}((1 \otimes m)\alpha(z))) \\ &= p(\Lambda_{\varphi_N}(z) \otimes \widehat{\Gamma}_M(m)), \end{aligned}$$

since, as recalled in Definition-Proposition 6.3.8, the linear span of the $(1 \otimes m)\alpha(z)$ forms a σ-strong*-norm core for $\Lambda_{\varphi_{N \rtimes M}}$. By the lemmas 6.4.3 and 6.4.7,

$$\tilde{\Lambda}(\rho_\alpha((1 \otimes m)\alpha(z))) = G(\Lambda_{\varphi_2}(\rho_\alpha((1 \otimes m)\alpha(z)))).$$

Since G is a left $N \rtimes M$-module map, we obtain that also

$$(\mathscr{L}^2(N_2), G^* \circ \tilde{\Lambda}, \pi_{N_2})$$

is a semi-cyclic representation for $\tilde{\varphi}_2$, and that $(G^* \circ \tilde{\Lambda}) \subseteq \Lambda_{\varphi_2}$.

By the first commutation relation of Lemma 6.4.5, it also follows that the modular operators for the semi-cyclic representations Λ_{φ_2} and $G^* \circ \tilde{\Lambda}$ are the same. Hence $\varphi_2 = \tilde{\varphi}_2$ by Proposition VIII.3.16 of [84].

□

Remark: This implies that T_2 equals $T_{\widehat{\alpha}} \circ (\rho_\alpha)_p^{-1}$ with $T_{\widehat{\alpha}}$ the canonical operator valued weight $N \rtimes M \to N$, by Theorem IX.4.18 of [84]. This generalizes Proposition 5.7 of [85] by removing the hypothesis that ρ_α is faithful.

It follows from Proposition 6.4.9 that G^* coincides with the map

$$Z : \mathscr{L}^2(N \rtimes M) \to \mathscr{L}^2(N_2)$$

which sends $\nu_M^{i/8} \Lambda_{\varphi_{N \rtimes M}}(z)$ to $\Lambda_{\tilde{\varphi}_2}(\rho_\alpha(z))$ for $z \in \mathscr{N}_{\varphi_{N \rtimes M}}$ (cf. the proof of Theorem 5.3 in [85]). So we can summarize our results by saying that the following natural square of $N \rtimes M$-bimodules and bimodule morphisms commutes:

$$\begin{array}{ccc}
\mathscr{L}^2(N_2) & \xrightarrow{\quad Z^* \quad} & \mathscr{L}^2(N \rtimes M) \\
\downarrow \cong & & \downarrow \cong \\
\mathscr{L}^2(N) \underset{\mu}{\otimes} \mathscr{L}^2(N) & \xrightarrow[G]{} & \mathscr{L}^2(N) \otimes \mathscr{L}^2(M)
\end{array} \tag{6.1}$$

Note that the above square was already constructed in the setting of algebraic quantum groups in [97].

For ease of reference, we write down explicitly how the bimodularity of G (or $\tilde{G}$, recalling the convention made in section 6.3) works on the two main parts of $N \rtimes M$.

Lemma 6.4.10. *Let α be an integrable coaction of a von Neumann algebraic quantum group M on a von Neumann algebra N. For all $x \in N$ and $m \in \widehat{M}'$, we have*

1. $\tilde{G}(x \underset{\mu}{\otimes} 1) = \alpha^{op}(x)\tilde{G}$,
2. $\tilde{G}(\hat{\pi}'_\alpha(m) \underset{\mu}{\otimes} 1) = (m \otimes 1)\tilde{G}$,
3. $\tilde{G}(1 \underset{\mu}{\otimes} \theta_N(x)) = (1 \otimes \theta_N(x))\tilde{G}$,
4. $\tilde{G}(1 \underset{\mu}{\otimes} \hat{\theta}'_\alpha(m)) = (\theta_{\widehat{M}'} \otimes \hat{\theta}'_\alpha)((\Delta_{\widehat{M}'})^{op}(m))\tilde{G}$.

Proof. As said, these equalities follow from the fact that G is a $N \rtimes M$-bimodule map. For the fourth one, we remark that the right representation $\theta_{N \rtimes M}$ of $N \rtimes M$ on $\mathscr{L}^2(N) \otimes \mathscr{L}^2(M)$ is given by

$$\theta_{N \rtimes M}(\alpha(x)) = \theta_N(x) \otimes 1$$

and

$$\theta_{N \rtimes M}(1 \otimes m) = U(1 \otimes \theta_{\widehat{M}'}(m))U^*,$$

a fact which is easy to recover using that $U = J_{N \rtimes M}(J_N \otimes J_{\widehat{M}})$. Now use that also $U = (\hat{\pi}'_\alpha \otimes \iota)(V_M)$, that V_M is the left regular multiplicative unitary for $\widehat{M}'$, and that $V_M(J_M \otimes J_{\widehat{M}}) = (J_M \otimes J_{\widehat{M}})V_M^*$. □

We introduce some more identities concerning modular automorphisms for integrable coactions. Recall from Proposition 6.3.15 that $\nabla^{it}_{\varphi_{N \rtimes M}} = \nabla^{it}_{\varphi_N} \otimes q^{it}_M$, using the notation of that proposition. Then $\kappa^M_t = q^{it}_M x q^{-it}_M$ defines a one-parametergroup of automorphisms on M, and $\gamma^{\widehat{M}'}_t(x) = q^{it}_M x q^{-it}_M$ defines a one-parametergroup of automorphisms on $\widehat{M}'$.

Lemma 6.4.11. *Let α be an integrable right coaction of a von Neumann algebraic quantum group M on a von Neumann algebra N.*

1. *For $x \in N$, we have* $\alpha(\sigma^{\varphi_N}_t(x)) = (\sigma^{\varphi_N}_t \otimes \kappa^M_t)(\alpha(x))$.
2. *For $m \in \widehat{M}'$, we have* $\sigma^{\varphi_2}_t(\hat{\pi}'_\alpha(m)) = \hat{\pi}'_\alpha(\gamma^{\widehat{M}'}_t(m))$.

Proof. The first statement follows from the Proposition 6.3.9 and 6.3.15. The second statement follows from the Lemmas 6.4.5 and 6.4.10, since for $m \in \widehat{M}'$, we then have

$$\begin{aligned} \pi_{N_2}(\sigma_t^{\varphi_2}(\hat{\pi}'_\alpha(m))) &= \nabla_{\varphi_2}^{it}(\hat{\pi}'_\alpha(m) \underset{\mu}{\otimes} 1)\nabla_{\varphi_2}^{-it} \\ &= \nabla_{\varphi_2}^{it} G^*(1 \otimes m) G \nabla_{\varphi_2}^{-it} \\ &= G^*(1 \otimes \gamma_t^{\widehat{M}'}(m))G \\ &= \pi_{N_2}(\hat{\pi}'_\alpha(\gamma_t^{\widehat{M}'}(m))). \end{aligned}$$

□

In particular, $\sigma_t^{\varphi_2}(\hat{\theta}_\alpha(\delta_{\widehat{M}}^{-is})) = \hat{\theta}_\alpha(\delta_{\widehat{M}}^{-is})$ for each $s, t \in \mathbb{R}$, since an easy computation shows that each $C_{\widehat{M}}(\delta_{\widehat{M}}^{-is})$ is invariant under γ_t. Since $\sigma_t^{\varphi_2}$ is implemented by $\nabla_{\varphi_N}^{it}$ on $\mathscr{L}^2(N)$, we obtain:

Corollary 6.4.12. *The one-parametergroups $\nabla_{\varphi_N}^{it}$ and $\hat{\theta}_\alpha(\delta_{\widehat{M}}^{-is})$ commute.*

We denote the resulting one-parametergroup of unitaries by

$$P_{\varphi_N}^{it} = \nabla_{\varphi_N}^{it} \, \hat{\theta}_\alpha(\delta_{\widehat{M}}^{it}).$$

Proposition 6.4.13. *Let α be an integrable right coaction of a von Neumann algebraic quantum group M on a von Neumann algebra N. Then N is invariant under $Ad(P_{\varphi_N}^{it})$.*

Proof. We only have to show that N is invariant under $\mathrm{Ad}(\hat{\pi}'_\alpha(C_{\widehat{M}}(\delta_{\widehat{M}}^{it}))$. But for any group-like element $u \in \widehat{M}'$, we have, denoting by $\hat{\alpha}$ the dual coaction, that

$$\begin{aligned} \hat{\alpha}((1 \otimes u)\alpha(x)(1 \otimes u^*)) &= (1 \otimes u \otimes u)(\alpha(x) \otimes 1_{\widehat{M}'})(1 \otimes u^* \otimes u^*) \\ &= ((1 \otimes u)\alpha(x)(1 \otimes u^*)) \otimes 1_{\widehat{M}'} \end{aligned}$$

for $x \in N$, and so, by the biduality theorem (Definition-Proposition 6.3.7), we get $\hat{\pi}'_\alpha(u)x\hat{\pi}'_\alpha(u)^* \in N$ after applying ρ_α. □

We denote the resulting one-parametergroup of automorphisms on N by

$$\tau_t^{\varphi_N} : N \to N : x \to P_{\varphi_N}^{it} x P_{\varphi_N}^{-it}.$$

Proposition 6.4.14. *Let α be an integrable right coaction of a von Neumann algebraic quantum group M on a von Neumann algebra N. Then the following identities hold for $x \in N$:*

$$\alpha(\tau_t^{\varphi_N}(x)) = (\tau_t^{\varphi_N} \otimes \tau_t^M)(\alpha(x)),$$

$$\alpha(\tau_t^{\varphi_N}(x)) = (\sigma_t^{\varphi_N} \otimes \sigma_{-t}^M)(\alpha(x)),$$

$$\alpha(\sigma_t^{\varphi_N}(x)) = (\tau_t^{\varphi_N} \otimes \sigma_t^M)(\alpha(x)).$$

Proof. By Lemma 6.4.11, we have

$$\alpha \circ \sigma_t^{\varphi_N} = (\sigma_t^{\varphi_N} \otimes \mathrm{Ad}(\delta_M^{-it})\tau_{-t}^M) \circ \alpha.$$

Further, we have

$$\alpha(\mathrm{Ad}(\widehat{\theta}_\alpha(\delta_{\widehat{M}}^{it}))(x)) = (\iota \otimes \mathrm{Ad}(C_{\widehat{M}}(\delta_{\widehat{M}}^{it})))(\alpha(x))$$

for $x \in N$, by the proof of the previous proposition. Now by the first formula of Theorem 4.17 in [92], we have $(J_{\widehat{M}}\delta_{\widehat{M}}^{-it}J_{\widehat{M}})P_M^{-it} = \nabla_M^{-it}$. So $\mathrm{Ad}(\delta_M^{-it})\tau_{-t}^M\mathrm{Ad}(J_{\widehat{M}}\delta_{\widehat{M}}^{-it}J_{\widehat{M}})$ reduces to σ_{-t}^M on M. This proves the second formula.

As for the first identity, we have, using the second identity, the coaction property of α and the identity $\Delta_M \circ \sigma_{-t}^M = (\sigma_{-t}^M \otimes \tau_t^M) \circ \Delta_M$, that

$$\begin{aligned}(\alpha \otimes \iota) \circ (\tau_t^{\varphi_N} \otimes \tau_t^M) \circ \alpha &= (\sigma_t^{\varphi_N} \otimes \sigma_{-t}^M \otimes \tau_t^M) \circ (\iota \otimes \Delta_M) \circ \alpha \\ &= (\iota \otimes \Delta_M) \circ (\sigma_t^{\varphi_N} \otimes \sigma_{-t}^M) \circ \alpha \\ &= (\alpha \otimes \iota) \circ \alpha \circ \tau_t^{\varphi_N}.\end{aligned}$$

Thus the first identity follows by the injectivity of α.

The third identity now easily follows from the first identity, the fact that $\mathrm{Ad}(C_{\widehat{M}}(\delta_{\widehat{M}}^{-it})))(m) = (\sigma_t^M\tau_{-t}^M)(m)$ for $m \in M$ (which again follows from $(J_{\widehat{M}}\delta_{\widehat{M}}^{-it}J_{\widehat{M}})P_M^{-it} = \nabla_M^{-it}$), and again the identity

$$\alpha(\mathrm{Ad}(\widehat{\theta}_\alpha(\delta_{\widehat{M}}^{it}))(x)) = (\iota \otimes \mathrm{Ad}(C_{\widehat{M}}(\delta_{\widehat{M}}^{it})))(\alpha(x))$$

for $x \in N$.

□

Lemma 6.4.15. *The one-parametergroup* $\tau_t^{\varphi_N}$ *satisfies* $\varphi_N \circ \tau_t^{\varphi_N} = \nu_M^t \varphi_N$, *and if* $x \in \mathscr{N}_{\varphi_N}$, *then*

$$P_{\varphi_N}^{it} \Lambda_{\varphi_N}(x) = \nu_M^{t/2} \Lambda_{\varphi_N}(\tau_t^{\varphi_N}(x)).$$

Proof. The first statement easily follows since

$$\begin{aligned} \varphi_N \circ \tau_t^{\varphi_N} &= \mu \circ (\iota_N \otimes \varphi_M) \circ \alpha \circ \tau_t^{\varphi_N} \\ &= \mu \circ \sigma_t^{\varphi_N} \circ (\iota_N \otimes \varphi_M \sigma_{-t}^M) \circ \alpha \\ &= \nu_M^{it}\, \mu \circ \sigma_t^{\mu} \circ (\iota_N \otimes \varphi_M) \circ \alpha \\ &= \nu_M^{it} \varphi_N, \end{aligned}$$

using the identities of the previous proposition.

By the first statement, $\mathrm{Ad}(\widehat{\theta}_\alpha(\delta_{\widehat{M}}^{-it}))(x) \in \mathscr{N}_{\varphi_N}$ for $x \in \mathscr{N}_{\varphi_N}$, and the second statement is equivalent with

$$\nu_M^{t/2} \widehat{\theta}_\alpha(\delta_{\widehat{M}}^{-it}) \Lambda_{\varphi_N}(x) = \Lambda_{\varphi_N}(\mathrm{Ad}(\widehat{\theta}_\alpha(\delta_{\widehat{M}}^{-it}))(x)),$$

where we remark that the right hand side also defines already a one-parametergroup of unitaries.

Now taking $x, y \in \mathscr{T}_{\varphi_N, T_\alpha}$, we have

$$G(\widehat{\theta}_\alpha(\delta_{\widehat{M}}^{-it}) \Lambda_{\varphi_N}(x) \underset{\mu}{\otimes} \Lambda_{\varphi_N}(y)) = (1 \otimes \theta_{\widehat{M}}(\delta_{\widehat{M}}^{-it}))(\Lambda_{\varphi_N} \otimes \Lambda_M)(\alpha(x)(y \otimes 1)).$$

Since $J_{\widehat{M}} \delta_{\widehat{M}}^{it} J_{\widehat{M}} = \nabla_M^{it} P_M^{-it}$ and

$$\alpha(\mathrm{Ad}(\widehat{\theta}_\alpha(\delta_{\widehat{M}}^{-it}))(x)) = (\iota \otimes \mathrm{Ad}(\theta_{\widehat{M}}(\delta_{\widehat{M}}^{-it}))(\alpha(x)),$$

we get that

$$G((\nu_M^{t/2} \widehat{\theta}_\alpha(\delta_{\widehat{M}}^{-it}) \Lambda_{\varphi_N}(x)) \underset{\mu}{\otimes} \Lambda_{\varphi_N}(y)) = G(\Lambda_{\varphi_N}(\mathrm{Ad}(\widehat{\theta}_\alpha(\delta_{\widehat{M}}^{-it}))(x)) \underset{\mu}{\otimes} \Lambda_{\varphi_N}(y)).$$

Since G is an isometry, and x, y were arbitrary, the result follows. □

6.5 Closed quantum subgroups

The following definition is taken from [89], Definition 2.9.

Definition 6.5.1. *Let M and M_1 be von Neumann algebraic quantum groups, and*

$$F : M_1 \to M$$

*a unital normal *-homomorphism. We say that* (M_1, F) is a closed quantum subgroup of M *when F is faithful, and*

$$(F \otimes F) \circ \Delta_{M_1} = \Delta_M \circ F.$$

The *closedness* in the foregoing definition is why we can define this concept on the von Neumann algebraic level (see the discussion after Definition 2.9 of [89]). The *general* notion of a quantum subgroup is treated in [54].

When convenient, we identify M_1 with its image $F(M_1)$, and we then just say that M_1 is a closed quantum subgroup of M.

Associated to a pair consisting of a von Neumann algebraic quantum group and a closed quantum subgroup, there are two coactions of the dual of the smaller one on the dual of the bigger one, resp. by 'left and right translation'.

Proposition 6.5.2. *Let $(\widehat{M_1}, \widehat{F})$ be a closed quantum subgroup of a von Neumann algebraic quantum group $\widehat{M}$. Then there is an integrable left coaction*

$$\gamma : M \to M_1 \otimes M$$

of M_1 on M, given by

$$\gamma(x) = W_{\widehat{F}}^*(1 \otimes x)W_{\widehat{F}} \in B(\mathscr{L}^2(M_1) \otimes \mathscr{L}^2(M)),$$

where $W_{\widehat{F}} := (\iota_{M_1} \otimes \widehat{F})(W_{M_1})$ coincides with the unitary implementation of γ. Furthermore, the left coaction γ_M commutes with the right coaction Δ_M of M on itself.

There also is an integrable right coaction

$$\alpha : M \to M \otimes M_1$$

of M_1 on M, given by

$$\alpha(x) = V_{\widehat{F}}(x \otimes 1)V_{\widehat{F}}^* \in B(\mathscr{L}^2(M) \otimes \mathscr{L}^2(M_1)),$$

where $V_{\widehat{F}} := ((C_{\widehat{M}} \circ \widehat{F} \circ C_{\widehat{M_1}}^{-1}) \otimes \iota_{M_1})(V_{M_1})$ coincides with the unitary implementation of α. This right coaction commutes with the left coaction Δ_M of M on itself.

Proof. We only sketch the proof for the right coaction. We have

$$(\Delta_{\widehat{M}'} \otimes \iota_{M_1})(V_{\widehat{F}}) = (V_{\widehat{F}})_{13}(V_{\widehat{F}})_{23},$$

so

$$(V_M)^*_{12}(V_{\widehat{F}})_{23}(V_M)_{12} = (V_{\widehat{F}})_{13}(V_{\widehat{F}})_{23}$$

and

$$(V_{\widehat{F}})_{23}(V_M)_{12}(V_{\widehat{F}})^*_{23} = (V_M)_{12}(V_{\widehat{F}})_{13}.$$

Since M is generated by the second leg of V_M, we have that α, as defined in the proposition, has range in $M \otimes M_1$. Since also $V_{\widehat{F}}$ is clearly a unitary corepresentation, α is a coaction.

Further, the stated equalities also imply that for $x \in M$, we have

$$\begin{aligned}(\Delta_M \otimes \iota_{M_1})\alpha(x) &= (V_M)_{12}(V_{\widehat{F}})_{13}(V_{\widehat{F}})_{23}(x \otimes 1 \otimes 1)(V_{\widehat{F}})^*_{23}(V_{\widehat{F}})^*_{13}(V_M)^*_{12} \\ &= (V_{\widehat{F}})_{23}(V_M)_{12}(x \otimes 1 \otimes 1)(V_M)^*_{12}(V_{\widehat{F}})^*_{23} \\ &= (\iota_M \otimes \alpha)\Delta_M(x).\end{aligned}$$

So we are in the situation stated after Proposition 12.1 of [54]. By Proposition 12.2 of [54], ψ_M is an α-invariant nsf weight on M, and then an adaptation of the argument in Proposition 4.3 of [85] shows that the unitary implementation U of α is given by

$$U(\Lambda_{\psi_M}(x) \otimes \Lambda_{\varphi_{M_1}}(y)) = (\Lambda_{\psi_M} \otimes \Lambda_{\varphi_{M_1}})(\alpha(x)(1 \otimes y)),$$

with $x \in \mathscr{N}_{\psi_M}$ and $y \in \mathscr{N}_{\varphi_{M_1}}$.

Now choose $\omega \in M_*$ such that $\omega(\,\cdot\,\delta_M^{1/2})$ extends from the space of left multipliers of $\delta_M^{1/2}$ to a normal functional ω_δ on M. Then

$$\begin{aligned}&((\omega \otimes \iota_{\widehat{M}})(W_M^*) \otimes 1)U(\Lambda_{\psi_M}(x) \otimes \Lambda_{\varphi_{M_1}}(y)) \\ &\quad= (\Lambda_{\psi_M} \otimes \Lambda_{\varphi_{M_1}})(((\omega_\delta \otimes \iota_M \otimes \iota_{M_1})(\Delta_M \otimes \iota_{M_1})\alpha(x))(1 \otimes y)) \\ &\quad= (\Lambda_{\psi_M} \otimes \Lambda_{\varphi_{M_1}})(\alpha((\omega_\delta \otimes \iota_M)\Delta_M(x))(1 \otimes y)) \\ &\quad= U((\omega \otimes \iota_{\widehat{M}})(W_M^*) \otimes 1)(\Lambda_{\psi_M}(x) \otimes \Lambda_{\varphi_{M_1}}(y)),\end{aligned}$$

which implies $U \in \widehat{M}' \otimes M_1$. A similar calculation also shows that

$$(V_M)_{12}U_{13}U_{23} = U_{23}(V_M)_{12},$$

which implies that

$$(\Delta_{\widehat{M}'} \otimes \iota_M)(U) = U_{13}U_{23}$$

since V_M is the left regular multiplicative unitary of $\widehat{M}'$.

Now since both U and $V_{\widehat{F}}$ implement α, the first leg of $U^*V_{\widehat{F}}$ lies in $\widehat{M}' \cap M' = \mathbb{C} \cdot 1_{B(\mathscr{L}^2(M))}$. Hence there exists $u \in M_1$ with

$$U = V_{\widehat{F}}(1 \otimes u).$$

But then the last equation in the previous paragraph, combined with the fact that $\widehat{F}$ preserves the comultiplication and that $(\Delta_{\widehat{M}'} \otimes \iota_M)(V_M) = (V_M)_{13}(V_M)_{23}$, implies that

$$(V_{\widehat{F}})_{13}(V_{\widehat{F}})_{23}(1 \otimes 1 \otimes u) = (V_{\widehat{F}})_{13}(1 \otimes 1 \otimes u)(V_{\widehat{F}})_{23}(1 \otimes 1 \otimes u),$$

which shows $u = 1$. Hence $U = V_{\widehat{F}}$.

□

One further has that the above coactions α and γ are related by the formula

$$\alpha \circ R_M = (R_M \otimes R_{M_1})\gamma^{\mathrm{op}}.$$

Now let $\widehat{M_1}$ be a closed quantum subgroup of the von Neumann algebraic quantum group $\widehat{M}$. Let α_M be the associated right coaction of M_1 on M. Suppose α_N is a right coaction of M on a von Neumann algebra N. Then $(\iota_N \otimes \alpha_M)\alpha_N(N) \subseteq \alpha_N(N) \otimes M_1$, for by Proposition 6.3.5, we only have to observe that the maps $(\iota_N \otimes \Delta_M \otimes \iota_{M_1})$ and $(\alpha_N \otimes \iota_M \otimes \iota_{M_1})$ coincide on the range of $(\iota_N \otimes \alpha_M)\alpha_N$, which follows by the equivariance of α_M and the fact that α_N is a coaction. Then it is easily seen that

$$\alpha_{N,1} := (\alpha_N^{-1} \otimes \iota_{M_1}) \circ (\iota_N \otimes \alpha_M)\alpha_N$$

defines a right coaction of M_1 on N.

Definition 6.5.3. *In the above situation, we call $\alpha_{N,1}$ the* restriction *of α_N to M_1.*

Similarly, one can restrict unitary right corepresentations. For this, we recall that unitary right corepresentations for a von Neumann algebraic quantum group M_1 are in one-to-one correspondence with non-degenerate right *-representations of $\widehat{A}_1^u$, the universal C*-algebra associated with its dual. Then if $\widehat{M_1} \subseteq \widehat{M}$ is a closed quantum subgroup, we also have a non-degenerate inclusion $\widehat{A}_1^u \subseteq \widehat{A}^u$, and then the restriction of a unitary right corepresentation U of M to M_1 is defined to be the unitary right corepresentation U_1 of M_1 corresponding to the restriction of the associated right *-representation of $\widehat{A}^u$ to $\widehat{A}_1^u$. It is not hard to see that, alternatively, U_1 is characterized by the fact that $(\iota \otimes \gamma_M)(U) = U_{1,12}U_{13}$ (or also $(\iota \otimes \alpha_M)(U) = U_{12}U_{1,13}$), where γ_M and α_M are the canonical left, resp. right coaction of M_1 on M.

Since coactions come with canonically associated corepresentations, it is a natural question to ask if the restriction process preserves this correspondence. This is answered by the following lemma.

Lemma 6.5.4. *Let $\widehat{M_1} \subseteq \widehat{M}$ be an inclusion of von Neumann algebraic quantum groups. Let α_N be a right coaction of M on a von Neumann algebra N, and U its canonical unitary implementation. Let $\alpha_{N,1}$ be the restriction of α_N to M_1, and $\tilde{U}_1$ the restriction of U to M_1. Then $\tilde{U}_1$ is the canonical unitary implementation of $\alpha_{N,1}$.*

Proof. Unfortunately, there seems to be no alternative but to follow again *from the start* the strategy of the proof of 'U is a corepresentation' from [85], Theorem 4.4. Indeed: Let $\tilde{U}_1$ be the restriction of U to M_1. Then we have seen that $(\iota \otimes \gamma_M)(U) = \tilde{U}_{1,12}U_{13}$. Hence we only have to show that the same identity holds with $\tilde{U}_1$ replaced by U_1, the canonical implementation of $\alpha_{N,1}$. Since the full proof would require considerable overlap with [85], we will only sketch how the procedure should be adapted to our situation.

To make the comparison with [85] more straightforward, we will henceforth work in the setting of left coactions and unitary left corepresentations. That is, we now suppose that we have a left coaction γ_N of M on a von Neumann algebra N, with canonical unitary implementation U (in the sense of [85]). We denote by $\gamma_{N,1}$ the restriction of γ_N to M_1, and by $\tilde{U}_1$ the restriction of U to M_1. We denote by U_1 the canonical implementation of $\gamma_{N,1}$. We want to show that[5]

$$(\alpha_M \otimes \iota)(U) = U_{1,23}U_{13}. \tag{6.2}$$

Suppose that Y is an arbitrary von Neumann algebra, and let

$$\gamma_{N \otimes Y} := \gamma_N \otimes \iota_Y : N \otimes Y \to M \otimes N \otimes Y$$

by the amplified coaction of γ_N. Then as in Theorem 4.4 of [85], it is easy to see that the unitary implementation of $\gamma_{N\otimes Y}$ on $\mathscr{L}^2(M) \otimes \mathscr{L}^2(N) \otimes \mathscr{L}^2(Y)$

[5] In [85], there is a difference in convention concerning what a left corepresentation is, and rather U^* is a left corepresentation in our terminology. Hence the change of order in the equation (6.2).

equals $U \otimes 1_{B(\mathscr{L}^2(Y))}$. Since restricting and amplifying obviously commute, we can thus replace γ_N by $\gamma_{N\otimes B(\mathscr{L}^2(M))}$.

Now by [85], $\gamma_{N\otimes B(\mathscr{L}^2(M))}$ is cocycle equivalent with a bidual coaction. Hence we should show that the equality (6.2) holds for integrable coactions γ_N, and that if it holds for one coaction, it also holds for any cocycle equivalent coaction.

We first prove the latter stability property. Let $\mathscr{V} \in M \otimes N$ be a γ_N-cocycle, and β_N the cocycle perturbation of γ_N by $\mathscr{V}$. Note first that $(\iota_{M_1}\otimes\Delta_M\otimes\iota_N)$ or $(\iota_{M_1} \otimes \iota_M \otimes \gamma_N)$ applied to $\mathscr{V}_{23}^*(\gamma_M \otimes \iota_N)(\mathscr{V})$ produces the same element. By Proposition 6.3.5, there exists $\mathscr{V}_1 \in M_1 \otimes N$ such that

$$(\gamma_M \otimes \iota_N)(\mathscr{V}) = \mathscr{V}_{23}(\iota_{M_1} \otimes \gamma_N)(\mathscr{V}_1).$$

Some further calculations then reveal that $\mathscr{V}_1$ is a 1-cocycle for $\gamma_{N,1}$, that the restriction $\beta_{N,1}$ of β_N to M_1 is precisely the cocycle perturbation of $\gamma_{N,1}$ with respect to $\mathscr{V}_1$, and that also

$$(\alpha_M \otimes \iota_N)(\mathscr{V}) = \mathscr{V}_{1,23}(\iota \otimes \gamma_{N,1})(\mathscr{V}).$$

Then Proposition 4.2 of [85], together with the final calculation appearing in that proof, show that the equality (6.2) holds for the unitary corepresentation associated with γ_N iff it holds for the one associated to β_N.

We now suppose that γ_N is integrable. First of all, note that there exists a strictly positive $d_{M_1}\eta M_1$ such that $\gamma_M(\delta_M^{it}) = d_{M_1}^{it} \otimes \delta_M^{it}$. In fact, $d_{M_1}^{it}$ is just the restriction of the unitary one-dimensional corepresentation δ_M^{it} to M_1. So each $d_{M_1}^{it}$ is a group-like element, hence invariant under $\tau_s^{M_1}$ and satisfying $R_{M_1}(d_{M_1}^{it}) = d_{M_1}^{-it}$. Now let μ be an nsf weight on N^{γ_N}, and put $\psi_N = \mu \circ T_{\gamma_N}$ where $T_{\gamma_N} = (\psi_M \otimes \iota_N)\gamma_N$. One checks that $\tilde{U}_1 \in M_1 \otimes B(\mathscr{L}^2(N))$ satisfies

$$(\omega_{\xi,\eta} \otimes \iota)(\tilde{U}_1)\Lambda_{\psi_N}(x) = \Lambda_{\psi_N}((\omega_{d_{M_1}^{1/2}\xi,\eta} \otimes \iota)\gamma_{N_1}(x))$$

for $\xi \in \mathscr{D}(d_{M_1}^{1/2})$, $\eta \in \mathscr{L}^2(M_1)$ and $x \in \mathscr{N}_{\varphi_N}$. (Compare Proposition 2.4 of [85].)

The proof is finished once we have shown that $\tilde{U}_1$ is precisely the unitary implementation of $\gamma_{N,1}$. But reading the proof of Proposition 4.3 in [85], we see that the whole discussion still works for the coaction $\gamma_{N,1}$ of M_1, replacing however the $\delta_{M_1}^{-1}$-invariant weight θ there by the $d_{M_1}^{-1}$-invariant weight ψ_N, and replacing every occurence of δ_{M_1} by d_{M_1}. Indeed, since we have remarked that d_{M_1} is invariant under the scaling group and inversed under the unitary antipode, we have that $d_{M_1}^{it}$ commutes with $\nabla_{\widehat{M}_1}^{is}$ and $\mathcal{T}_{\widehat{M}}$, since these implement respectively the scaling group and the composition of the antipode with the *-operation. Also the commutation of $\tilde{U}_1$ with $d_{M_1}^{it}\nabla_{\widehat{M}_1}^{it} \otimes \nabla_{\psi_N}^{it}$ still holds true: this easily reduces to the identity

$$(\mathrm{Ad}(d_M^{it}) \otimes \iota_N) \circ (\tau_t^{M_1} \otimes \sigma_t^{\psi_N}) \circ \gamma_{N,1} = \gamma_{N,1} \circ \sigma_t^{\psi_N},$$

which in turn reduces to known identities by applying $(\iota \otimes \gamma_N)$. Since these two facts are the main ingredients which make Proposition 4.3 of [85] work, we are done.

□

One can also *induce* coactions from a smaller quantum group to a bigger one. Let again $\widehat{M}_1$ be a closed quantum subgroup of a von Neumann algebraic quantum group $\widehat{M}$. Let γ_M be the associated *left* coaction of M_1 on M, and suppose that we have a right coaction α_{N_1} of M_1 on a von Neumann algebra N_1. Then we can create a new right coaction $\alpha_N = \mathrm{Ind}_M(\alpha_{N_1})$ of M on the von Neumann algebra

$$N = \mathrm{Ind}_M(N_1) := \{z \in N_1 \otimes M \mid (\alpha_{N_1} \otimes \iota_M)z = (\iota_{N_1} \otimes \gamma_M)z\},$$

defined as

$$\alpha_N := (\iota_{N_1} \otimes \Delta_M).$$

Definition 6.5.5. *In the above situation, we call $\alpha_N = Ind_M(\alpha_{N_1})$ the* induced coaction (of α_{N_1} along M).

Lemma 6.5.6. *Let $\widehat{M}_1$ be a closed quantum subgroup of a von Neumann algebraic quantum group $\widehat{M}$. If α_{N_1} is a right coaction of M_1 on a von Neumann algebra N_1, and α_N is the induced coaction, then $N_1 \rtimes M_1$ is W*-Morita equivalent with $N \rtimes M$.*

Proof. This is implicit in [87], which however works completely in the C*-algebraic setting. We therefore only give a quick sketch of the proof.

Denote $\mathscr{H} = \mathscr{L}^2(N_1) \otimes \mathscr{L}^2(M)$. We will make $\mathscr{H}$ into an $N \rtimes M$-$N_1 \rtimes M_1$-equivalence correspondence. First note that by definition, $N \rtimes M$ is a von Neumann algebra contained in $N_1 \otimes (\Delta_M(M) \cup (1 \otimes \widehat{M}'))'' = N_1 \otimes (M \rtimes M)$, hence it has a natural faithful normal left representation on $\mathscr{H}$. Now denote by $\mathscr{J}$ the set of z in $N_1 \otimes B(\mathscr{L}^2(M_1), \mathscr{L}^2(M))$ such that

$$(\alpha_{N_1} \otimes \iota)(z) = (W^*_{M_1})_{23} z_{13} ((\iota \otimes \pi_{\hat{M}_1}) W_{M_1})_{23}.$$

Then a standard argument shows that the σ-weak closure of $\mathscr{J}\mathscr{J}^*$ coincides with all operators z in $N \otimes B(\mathscr{L}^2(M))$ for which

$$(\alpha_{N_1} \otimes \iota)(z) = (W^*_{M_1})_{23} z_{13} (W^*_{M_1})_{23}.$$

But this is easily identified with the image of $N \rtimes M$. Since the σ-weak closure of $\mathscr{J}^*\mathscr{J}$ is also seen to be exactly $N_1 \rtimes M_1$, we are done.

□

Chapter 7

Galois objects for von Neumann algebraic quantum groups

In this chapter, we examine those integrable coactions of a von Neumann algebraic quantum group M which are ergodic and have a faithful Galois homomorphism. We show that in this case, the von Neumann algebra N acted upon contains a 'modular element', which allows to create on N (and $\mathscr{L}^2(N)$) a structure which is very similar to the one of a von Neumann algebraic quantum group. We then show that with this structure available, we can turn $\widehat{P}$, by which we denote the commutant of the associated right representation of $\widehat{M}$ on $\mathscr{L}^2(N)$, into a von Neumann algebraic quantum group. We then provide some more information about the global structures connecting $\widehat{M}$, N and $\widehat{P}$, and in particular examine the associated C*-algebraic aspects.

7.1 Galois coactions

Definition 7.1.1. *Let N be a von Neumann algebra, M a von Neumann algebraic quantum group, and α an integrable coaction of M on N. We call α a* Galois coaction *if the Galois homomorphism ρ_α is faithful, or equivalently, if the Galois isometry $\tilde{G}$ is a unitary (in which case we call it the* Galois unitary*).*

We present some natural examples of Galois coactions.

Example 7.1.2. *Every dual coaction, or more generally, every semidual coaction is Galois.*

Proof. Recall from [85] that a semidual right coaction of a von Neumann algebraic quantum group M on a von Neumann algebra N is a right coaction α for which there exists a unitary $v \in N \otimes B(\mathscr{L}^2(M))$ such that

$$(\alpha \otimes \iota)(v) = v_{13}(W_M)_{23}$$

holds. Such coactions are Galois by Proposition 5.12 of [85]. □

Example 7.1.3. *Every integrable outer coaction is Galois.*

Proof. Recall again from [85] that a coaction α of a von Neumann algebraic quantum group M on a von Neumann algebra N is called outer when

$$N \rtimes M \cap \alpha(N)' = \mathbb{C} \cdot 1_N$$

holds. Thus an *integrable* outer coaction is automatically Galois, since $N \rtimes M$ is then a factor. □

Our next example shows that the natural 'quantum fibre bundle' structure associated to a quantum homogeneous space indeed comes from a Galois coaction (which seems a prime requisite for any theory generalizing the classical theory).

Example 7.1.4. *If M_1 and M are von Neumann algebraic quantum groups, and $(\widehat{M}_1, \widehat{F})$ a closed quantum subgroup of $\widehat{M}$, the associated right coaction α of M_1 on M is Galois. Conversely, if M and M_1 are von Neumann algebraic quantum groups for which there is a right Galois coaction α of M_1 on M, such that*

$$(\iota_M \otimes \alpha)\Delta_M = (\Delta_M \otimes \iota_{M_1})\alpha,$$

then $\widehat{M}_1$ can be made into a closed quantum subgroup of $\widehat{M}$, in such a way that α is precisely the coaction by right translations.

Proof. First suppose that $(\widehat{M}_1, \widehat{F})$ is a closed quantum subgroup of $\widehat{M}$. Denote $\widehat{F}' = C_{\widehat{M}} \circ \widehat{F} \circ C_{\widehat{M}_1}^{-1}$, and denote

$$V_{\widehat{F}} = (\widehat{F}' \otimes \iota_{M_1})(V_{M_1}).$$

Then we can make the following sequence of isomorphisms:

$$\begin{aligned} M \ltimes M_1 &= (\alpha(M) \cup (1 \otimes \widehat{M}_1'))'' \\ &\cong ((M \otimes 1) \cup V_{\hat{F}}^*(1 \otimes \widehat{M}_1')V_{\hat{F}})'' \\ &= ((M \otimes 1) \cup (\hat{F}' \otimes \iota)(\Delta_{\widehat{M}_1'}(\widehat{M}_1')))'' \\ &\cong ((M \otimes 1) \cup \Delta_{\widehat{M}'}(\hat{F}'(\widehat{M}_1')))'' \\ &\cong (M \cup \hat{F}'(\widehat{M}_1'))'', \end{aligned}$$

where we used that V_{M_1} is the left regular corepresentation for $(\widehat{M}_1', \Delta_{\widehat{M}_1'})$. Since it's easy to see that the resulting isomorphism satisfies the requirements for the Galois homomorphism (using that $V_{\hat{F}}$ is actually the unitary corepresentation implementing α, by Proposition 6.5.2), the coaction is Galois.

Now suppose that we have a Galois coaction α such that $(\iota_M \otimes \alpha)\Delta_M = (\Delta_M \otimes \iota_{M_1})\alpha$. Denote by $(\hat{A}_u', \Delta_{\hat{A}'u})$ the universal C*-algebraic quantum group associated with $\widehat{M}'$, and similarly for $\widehat{M}_1'$. Then just as in Proposition 6.5.2, the unitary implemntation U of α is determined by

$$U(\Lambda_{\psi_M}(x) \otimes \Lambda_{\varphi_{M_1}}(y)) = (\Lambda_{\psi_M} \otimes \Lambda_{\varphi_{M_1}})(\alpha(x)(1 \otimes y))$$

for $x \in \mathscr{N}_{\psi_M}$ and $y \in \mathscr{N}_{\varphi_{M_1}}$, and further $U \in \widehat{M}' \otimes M_1$ with $(\Delta_{\widehat{M}'} \otimes \iota_{M_1})(U) = U_{13}U_{23}$. From this, it is easy to conclude that $(\widehat{M}_1, \hat{\pi}_\alpha)$ is a closed quantum subgroup of $\widehat{M}$, using the concrete form of the implementation of $\hat{\pi}_\alpha$. Since also $U = (\hat{\pi}_\alpha' \otimes \iota_{M_1})(V_{M_1})$, we also get that α is precisely the coaction associated to the closed quantum subgroup $(\widehat{M}_1, \hat{\pi}_\alpha)$.

□

7.2 Structure of Galois objects

Our main object of study from now on will be the Galois coactions which have trivial coinvariants. We show that such coactions automatically have a (unique) invariant nsf weight, but our approach is different from the one for algebraic quantum groups: we will *first* search a 1-cocycle to deform φ_N (which will be of the form $\nu_M^{it^2/2}\delta_N^{it}$, for some non-singular positive operator $\delta_N \eta N$), and *then* show that this deformation is an invariant nsf weight on N.

Definition 7.2.1. *If N is a von Neumann algebra, M a von Neumann algebraic quantum group and α_N an* ergodic *Galois coaction of M on N, we call (N, α_N) a right* Galois object *for M.*

In the rest of this section, we suppose that we have fixed some right Galois object N for a von Neumann algebraic quantum group M. Then

$$T_{\alpha_N} = (\iota_N \otimes \varphi_M)\alpha_N$$

itself will already be an nsf weight on N, so we denote it by φ_N. Then $\mathscr{N}_{T_{\alpha_N}} = \mathscr{N}_{\varphi_N}$. We will from now on use a different notation for the standard GNS map: *whenever working with a Galois object, we will write Λ_N instead of Λ_{φ_N}.* We further denote $\sigma_t^{\varphi_N}$ as σ_t^N and $\nabla_{\varphi_N}^{it}$ as ∇_N^{it}. We keep denoting the unitary implementation of α_N by U.

For such a Galois object, $N \rtimes M \underset{\rho_{\alpha_N}}{\cong} N_2$ becomes the whole of $B(\mathscr{L}^2(N))$, and $\varphi_2 = \mathrm{Tr}(\,\cdot\,\nabla_N)$. Further, we can identify $\mathscr{L}^2(N_2)$ with $\mathscr{L}^2(N)\otimes\mathscr{L}^2(N)$ by the map

$$\Lambda_{\varphi_2}(\Lambda_N(x)\Lambda_N(y^*)^*) \to \Lambda_N(x) \otimes \Lambda_N(y) \qquad \text{for } x, y \in \mathscr{T}_{\varphi_N}.$$

For $x \in B(\mathscr{L}^2(N))$, we then have

$$\pi_{N_2}(x) = \pi_N(x) \otimes 1, \qquad \theta_{N_2}(x) = 1 \otimes \theta_N(x),$$

while the modular structure of N_2 is given by

$$\nabla_{N_2}^{it} = \nabla_N^{it} \otimes \nabla_N^{it}$$

and

$$J_{N_2} = \Sigma(J_N \otimes J_N).$$

We remark that now for $x, y \in \mathscr{N}_{\varphi_N}$, we have

$$\tilde{G}(\Lambda_N(x) \otimes \Lambda_N(y)) = \Sigma(\Lambda_N \otimes \Lambda_M)(\alpha_N(x)(y \otimes 1)),$$

by a simple argument, and then also

$$(\iota \otimes \omega)(\tilde{G})\Lambda_N(x) = \Lambda_M((\omega \otimes \iota_M)\alpha_N(x))$$

for all $x \in \mathscr{N}_{\varphi_N}$ and $\omega \in N_*$.

The following piece of extra structure on N has already been obtained (see the remark after Proposition 6.4.13).

Definition 7.2.2. *Let N be a right Galois object for a von Neumann algebraic quantum group M. We call the one-parametergroup*

$$\tau_t^{\varphi_N} : N \to N : x \to P_{\varphi_N}^{it} x P_{\varphi_N}^{-it}$$

the scaling group *of the Galois object N.*

We will denote $\tau_t^{\varphi_N}$ as τ_t^N, and P_{φ_N} as P_N.

We prove some statements concerning the Galois unitary $\tilde{G}$ for a Galois object N (see Definition 6.4.2 and the discussion just before it). Note that $\mathscr{L}^2(N)$ carries the right $\widehat{M}$-*-representation $\widehat{\theta}_{\alpha_N}$, and that we had denoted by $\widehat{Q} = \begin{pmatrix} \widehat{P} & \widehat{N} \\ \widehat{O} & \widehat{M} \end{pmatrix}$ the linking von Neumann algebra between the right $\widehat{M}$-modules $\mathscr{L}^2(M)$ and $\mathscr{L}^2(N)$.

Lemma 7.2.3. *1. $\tilde{G} \in \widehat{O} \otimes N$.*

2. $\tilde{G}_{12} U_{13} = (V_M)_{13} \tilde{G}_{12}$.

Proof. The first statement follows by the second and third commutation relation of Lemma 6.4.10. Since for $\omega \in M_*$, we have $(\iota \otimes \omega)(U) = \widehat{\pi}'_{\alpha_N}((\iota \otimes \omega)(V_M))$, the second statement also follows from the second commutation relation of Lemma 6.4.10. □

The following is just a restatement of Lemma 6.4.5.

Lemma 7.2.4. *The map $\tilde{G}$ satisfies the identity*

$$\tilde{G}(J_N \otimes J_N)\Sigma = \Sigma U \Sigma (J_{\widehat{M}} \otimes J_N)\tilde{G}.$$

Now we prove a pentagonal identity:

Proposition 7.2.5. *Let N be a right Galois object for a von Neumann algebraic quantum group M. Then*

$$(W_{\widehat{M}})_{12} \tilde{G}_{13} \tilde{G}_{23} = \tilde{G}_{23} \tilde{G}_{12}.$$

Proof. For $x \in \mathscr{N}_{\varphi_N}$ and $\omega \in B(\mathscr{L}^2(N))_*$, we have $(\omega \otimes \iota_M)(\alpha_N(x)) \in \mathscr{N}_{\varphi_M}$, by Lemma 6.3.10. Now consider ω of the form $\omega_{\Lambda_N(y),\Lambda_N(z)}$ with $y, z \in \mathscr{N}_{\varphi_N}$. Then

$$(\iota_{\widehat{O}} \otimes \omega)(\tilde{G})\Lambda_N(x) = \Lambda_M((\omega \otimes \iota_M)(\alpha_N(x))).$$

Using the closedness of the map Λ_M, we can conclude that the previous identity holds for all $\omega \in B(\mathscr{L}^2(N))_*$.

Now for $x \in \mathscr{N}_{\varphi_N}$, $\omega \in M_*$ and $\omega' \in N_*$, we have, using $W_{\widehat{M}} = \Sigma W_M^* \Sigma$,

$$\begin{aligned}
&(\iota_{\widehat{M}} \otimes \omega)(W_{\widehat{M}})(\iota_{\hat{O}} \otimes \omega')(\tilde{G})\Lambda_N(x) \\
&\quad= \Lambda_M((\omega' \otimes \omega \otimes \iota_M)((\iota_N \otimes \Delta_M)(\alpha_N(x)))) \\
&\quad= \Lambda_M((\omega' \otimes \omega \otimes \iota_M)((\alpha_N \otimes \iota_M)(\alpha_N(x)))) \\
&\quad= \Lambda_M((((\omega \otimes \omega') \circ \alpha_N^{\mathrm{op}}) \otimes \iota_M)(\alpha_N(x))) \\
&\quad= (\iota_{\hat{O}} \otimes ((\omega \otimes \omega') \circ \alpha_N^{\mathrm{op}}))(\tilde{G})\Lambda_N(x),
\end{aligned}$$

from which we conclude

$$(W_{\widehat{M}})_{12}\tilde{G}_{13} = (\iota_{\hat{O}} \otimes \alpha_N^{\mathrm{op}})(\tilde{G}).$$

Since

$$(\iota_{\hat{O}} \otimes \alpha_N^{\mathrm{op}})(\tilde{G}) = \tilde{G}_{23}\tilde{G}_{12}\tilde{G}_{23}^*$$

by the first commutation relation in Lemma 6.4.10, the result follows.

□

Remark: When N and M have infinite-dimensional separable preduals, then choosing a unitary $u : \mathscr{L}^2(M) \to \mathscr{L}^2(N)$, the unitary $v = \tilde{G}(u \otimes 1)$ in $B(\mathscr{L}^2(M)) \otimes N$ will satisfy $(\iota \otimes \alpha_N)(v) = (W_{\widehat{M}})_{13}v_{12}$. So in this case there is a one-to-one correspondence between Galois objects and ergodic semi-dual coactions. The same is true when either N or M are finite-dimensional, since in this case one can show that they then both have the same dimension.[1] However, we do not know of an easy argument showing that for general Galois objects, an orthonormal basis of $\mathscr{L}^2(N)$ has the same cardinality as one for $\mathscr{L}^2(M)$, which would make the previous statement true for *any* Galois object.

We have the following density results:

Lemma 7.2.6. *1. The space*

$$L = \{(\omega \otimes \iota_N)(\tilde{G}) \mid \omega \in B(\mathscr{L}^2(N), \mathscr{L}^2(M))_*\}$$

is σ-weakly dense in N.

[1] We do not know of a concrete reference for this fact, but it follows easily from the results in the tenth chapter.

2. The space

$$K = \{(\iota_{\hat{O}} \otimes \omega)(\tilde{G}) \mid \omega \in B(\mathscr{L}^2(N))_*\}$$

is σ-weakly dense in $\hat{O}$.

Proof. By the pentagonal identity for $\tilde{G}$ in Proposition 7.2.5, the linear span of the $(\omega \otimes \iota)(\tilde{G})$ will be an algebra. Further, for any $x \in \mathscr{N}_{\varphi_N}$ and $m \in \mathscr{N}_{\varphi_M}$, we have

$$(1 \otimes m^*)\alpha_N(x) \in \mathscr{M}_{(\iota \otimes \varphi_M)}$$

and

$$(\omega_{\Lambda_N(x),\Lambda(m)} \otimes \iota)(\tilde{G}) = (\iota \otimes \varphi_M)((1 \otimes m^*)\alpha_N(x)),$$

by an easy calculation. From this, we can conclude that the σ-weak closure of L coincides with the σ-weak closure of the span of

$$\{(\iota \otimes \omega)(\alpha_N(x)) \mid \omega \in M_*, x \in N\},$$

so that this σ-weak closure will be a unital sub-von Neumann algebra of N (see also the proof of Proposition 1.21 of [92]), which is known to be dense in N (a fact which holds for *any* coaction α). For completeness, we give a proof of this last fact. Suppose $\omega \in N_*$ is orthogonal to L. By the biduality theorem (see [32], and also Theorem 2.6 of [85]), we have that $(\alpha_N(N) \cup (1 \otimes B(\mathscr{L}^2(M))))''$ equals $N \otimes B(\mathscr{L}^2(M))$. So for any $x \in N \otimes B(\mathscr{L}^2(N))$ and $\omega' \in B(\mathscr{L}^2(N))_*$, $(\iota \otimes \omega')(x)$ can be σ-weakly approximated by elements of the form $(\iota \otimes \omega')(x_n)$ with x_n in the algebra generated by $\alpha_N(N)$ and $1 \otimes B(\mathscr{H})$, and any such element can in turn be approximated by an element in the algebra generated by elements of the form $(\iota \otimes \omega'')(\alpha_N(x_{nm}))$, $\omega'' \in B(\mathscr{L}^2(M))_*$ and $x_{nm} \in N$, by using an orthogonal basis argument. It follows that ω vanishes on the whole of N, and hence L is σ-weakly dense in N.

For the second statement, note that, again by the pentagonal identity for $\tilde{G}$, we have that K is closed under left multiplication with elements of the form $(\iota \otimes \omega)(W_{\widehat{M}})$ for $\omega \in M_*$. Hence, as in the proof of Proposition 5.7.3, it is enough to show that if $z \in \widehat{N}$ satisfies $K \cdot z = 0$, then $z = 0$. But take $x, y \in \mathscr{T}_{\varphi_N}$, and $m \in \mathscr{N}_{\varphi_{\widehat{M}'}}$. Then

$$(\iota \otimes \omega_{\Lambda_N(x),\Lambda_N(y)})(\tilde{G}^*)\widehat{\Gamma}_M(m) = \widehat{\pi}'_{\alpha_N}(m)x\Lambda_N(\sigma^N_{-i}(y^*))$$

by Lemma 5.7.7 and Lemma 6.4.3. Hence $K^* \cdot \mathscr{L}^2(M)$ is dense in $\mathscr{L}^2(N)$, and necessarily $z = 0$. ☐

Proposition 7.2.7. *We have the following commutation relations:*

1. $\tilde{G}(\nabla_N^{it} \otimes \nabla_N^{it}) = (\delta_M^{-it}\nabla_{\widehat{M}}^{-it} \otimes \nabla_N^{it})\tilde{G}$,
2. $\tilde{G}(\nabla_N^{it} \otimes P_N^{it}) = (\nabla_M^{it} \otimes P_N^{it})\tilde{G}$,
3. $\tilde{G}(P_N^{it} \otimes P_N^{it}) = (P_M^{it} \otimes P_N^{it})\tilde{G}$.

Proof. The first identity follows immediately from Lemma 6.4.5 and Proposition 6.3.15, while the other two follow by using the definition of G, the implementation of Lemma 6.4.15 and the identities in Lemma 6.4.14. □

Lemma 7.2.8. *For any $m \in M'$, the operator $\tilde{G}^*(m \otimes 1)\tilde{G}$ lies in $N' \otimes N$.*

Proof. Clearly, the second leg lies in N. Since $\tilde{G}(y \otimes 1)\tilde{G}^* = \alpha_N^{\mathrm{op}}(y)$ for $y \in N$, the first leg of $\tilde{G}^*(m \otimes 1)\tilde{G}$ must be inside N'. □

Remark: For general Galois coactions α, this lemma is still true if we replace $N' \otimes N$ by $\pi_l^{N_2}(N)' \cap N_3$, where N_3 is the next step in the tower construction:

$$N^\alpha \subseteq N \subseteq N_2 \subseteq N_3,$$

where N_3 is precisely $B(\mathscr{L}^2(N)) \otimes N$ in case of a Galois object. However, it is the degeneracy of $\pi_l^{N_2}(N)' \cap N_3$, i.e. the fact that it can be written as an ordinary tensor product, which allows us to continue.

Consider $H^{it} = \tilde{G}^*(J_M \delta_M^{it} J_M \otimes 1)\tilde{G}$ in $N' \otimes N$.

Lemma 7.2.9. *There exist non-singular positive h, k affiliated with respectively N' and N such that $H^{it} = h^{it} \otimes k^{it}$ for all $t \in \mathbb{R}$. Moreover, we have $\alpha_N(k^{it}) = k^{it} \otimes \delta_M^{it}$ for $t \in \mathbb{R}$.*

Proof. We show that $H^{it}(B(\mathscr{L}^2(N)) \otimes 1)H^{-it} = B(\mathscr{L}^2(N)) \otimes 1$. Since $B(\mathscr{L}^2(N)) = \rho_{\alpha_N}(N \rtimes M)$, we only have to show that $H^{it}(N \otimes 1)H^{-it} = (N \otimes 1)$ and $H^{it}(\hat{\pi}'_{\alpha_N}(\widehat{M}') \otimes 1)H^{-it} = (\hat{\pi}'_{\alpha_N}(\widehat{M}') \otimes 1)$. Now the first equality is clear as the first leg of H^{it} lies in N'. As for the second equality, applying $\tilde{G}(\,\cdot\,)\tilde{G}^*$, this is equivalent with $\mathrm{Ad}(J_M \delta_M^{it} J_M)(\widehat{M}') = \widehat{M}'$, which is easily seen to be true.

Denote by h a positive operator which implements the automorphism group $\mathrm{Ad}(H^{it})$ on $B(\mathscr{L}^2(N))$, so

$$\mathrm{Ad}(H^{it})(x \otimes 1) = (\mathrm{Ad}(h^{it})(x)) \otimes 1$$

for all $x \in B(\mathscr{L}^2(N))$. (This is well-known to be possible for $\mathscr{L}^2(N)$ separable (see e.g. Theorem XI.3.11 of [84]). An easy maximality argument shows that, in this case, it holds regardless of separability.) Then h is non-singular, with h affiliated with N', and $H^{it} = h^{it} \otimes k^{it}$ for a positive non-singular k affiliated with N.

Note now that $W^*_{\widehat{M}}(J_M\delta_M^{it}J_M \otimes 1)W_{\widehat{M}} = J_M\delta_M^{it}J_M \otimes \delta_M^{it}$, which can be computed for example by Lemma 4.14 and the formulas in Proposition 4.17 of [92]. Then using the pentagonal identity for $\tilde{G}$, we have

$$\begin{aligned}
(\iota \otimes \alpha_N^{\mathrm{op}})(H^{it}) &= \tilde{G}_{23}H^{it}_{12}\tilde{G}^*_{23} \\
&= \tilde{G}_{23}\tilde{G}^*_{12}(J_M\delta_M^{it}J_M \otimes 1 \otimes 1)\tilde{G}_{12}\tilde{G}^*_{23} \\
&= \tilde{G}^*_{13}(W_{\widehat{M}})^*_{12}\tilde{G}_{23}(J_M\delta_M^{it}J_M \otimes 1 \otimes 1)\tilde{G}^*_{23}(W_{\widehat{M}})_{12}\tilde{G}_{13} \\
&= \tilde{G}^*_{13}(J_M\delta_M^{it}J_M \otimes \delta_M^{it} \otimes 1)\tilde{G}_{13} \\
&= h^{it} \otimes \delta_M^{it} \otimes k^{it},
\end{aligned}$$

so that $\alpha_N(k^{it}) = k^{it} \otimes \delta_M^{it}$.

□

The operator k which appears in the lemma is determined up to a positive scalar. *We will now fix some k, and denote it as δ_N.*

Definition 7.2.10. *We call δ_N the* modular element *of the Galois object N.*

Lemma 7.2.11. *With the notation of the previous lemma, we have*

1. $h = J_N\delta_N^{-1}J_N$,
2. $\sigma_t^N(\delta_N^{is}) = \nu_M^{ist}\delta_N^{is}$,
3. $\tau_t^N(\delta_N^{is}) = \delta_N^{is}$.

Proof. Denoting again $H^{it} = \tilde{G}^*(J_M\delta_M^{it}J_M \otimes 1)\tilde{G}$, we first prove that

$$\Sigma(J_N \otimes J_N)H^{it}(J_N \otimes J_N)\Sigma = H^{it}.$$

Using Lemma 7.2.4, the left hand side equals

$$\tilde{G}^*(J_{\widehat{M}} \otimes J_N)\Sigma U^*\Sigma(J_M\delta_M^{it}J_M \otimes 1)\Sigma U\Sigma(J_{\widehat{M}} \otimes J_N)\tilde{G}.$$

As $U \in B(\mathscr{L}^2(N)) \otimes M$, this reduces to $\tilde{G}^*(J_{\widehat{M}}J_M\delta_M^{it}J_MJ_{\widehat{M}} \otimes 1)\tilde{G}$. Since J_M commutes with $J_{\widehat{M}}$ up to a scalar of modulus 1, and since δ_M^{it} commutes with $J_{\widehat{M}}$, we find that this expression reduces to $\tilde{G}^*(J_M\delta_M^{it}J_M\otimes 1)\tilde{G} = H^{it}$. So

$$J_N\delta_N^{it}J_N\otimes J_Nh^{it}J_N = h^{it}\otimes\delta_N^{it},$$

which implies that there exists a positive scalar r such that $h^{it} = r^{it}J_N\delta_N^{it}J_N$. But plugging this back into the above equality, we find that $r^{2it} = 1$ for all t, hence $r = 1$.

For the second statement, we easily get, using the first commutation relation of Proposition 7.2.7, that

$$(\nabla_N^{it}\otimes\nabla_N^{it})(J_N\delta_N^{is}J_N\otimes\delta_N^{is})(\nabla_N^{-it}\otimes\nabla_N^{-it}) = (J_N\delta_N^{is}J_N\otimes\delta_N^{is}).$$

This implies that there exists a positive number ν_N such that $\sigma_t^N(\delta_N^{is}) = \nu_N^{ist}\delta_N^{is}$. We must show that $\nu_N = \nu_M$.

But we know now that δ_N^{is} is analytic with respect to σ_t^N. So if $x\in\mathscr{M}_{\varphi_N}$, then also $x\delta_N^{is}$ and $\delta_N^{is}x$ are integrable. We have for such x that, choosing some state $\omega\in N_*$,

$$\begin{aligned}
\varphi_N(\delta_N^{is}x) &= \varphi_M((\omega\otimes\iota)(\alpha_N(\delta_N^{is}x)))\\
&= \varphi_M(\delta_M^{is}(\omega(\delta_N^{is}\,\cdot\,)\otimes\iota)(\alpha_N(x)))\\
&= \nu_M^s\varphi_M((\omega(\delta_N^{is}\,\cdot\,)\otimes\iota)(\alpha_N(x))\delta_M^{is})\\
&= \nu_M^s\varphi_M((\omega(\delta_N^{is}\,\cdot\,\delta_N^{-is})\otimes\iota)(\alpha_N(x\delta_N^{is})))\\
&= \nu_M^s\varphi_N(x\delta_N^{is}).
\end{aligned}$$

This shows $\sigma_{-i}^N(\delta_N^{is}) = \nu_M^s\delta_N^{is}$, which implies $\nu_N = \nu_M$.

As for the last statement, this follows from

$$\begin{aligned}
\alpha_N(\tau_t^N\sigma_{-t}^N(\delta_N^{is})) &= (\iota\otimes\tau_t^M\sigma_{-t}^M)\alpha_N(\delta_N^{is})\\
&= \delta_N^{is}\otimes\tau_t^M\sigma_{-t}^M(\delta_M^{is})\\
&= \nu_M^{-ist}\alpha_N(\delta_N^{is}).
\end{aligned}$$

□

By Connes' cocycle derivative theorem (Theorem 5.2.8), we can now make the nsf weight $\psi_N := \varphi_N(\delta_N^{1/2}\,\cdot\,\delta_N^{1/2})$, by which we mean the deformation of φ_N by the 1-cocycle $w_t = \nu_M^{it^2/2}\delta_N^{it}$.

Theorem 7.2.12. *Let N be a right Galois object for a von Neumann algebraic quantum group M. Then the weight ψ_N is invariant with respect to α_N.*

Proof. Let $x \in N$ be a left multiplier of $\delta_N^{1/2}$ such that $x\delta_N^{1/2}$ is an element of $\mathscr{N}_{\varphi_N}$. Then $x \in \mathscr{N}_{\psi_N}$, and there is a unique semi-cyclic representation Γ_N for ψ_N in $\mathscr{L}^2(N)$ such that $\Gamma_N(x) = \Lambda_N(x\delta_N^{1/2})$ for all such x (see the remark before Proposition 1.15 in [56]). Choose $\xi \in \mathscr{D}(\delta_M^{-1/2})$. Then for any $\eta \in \mathscr{L}^2(M)$, we have $(\iota \otimes \omega_{\xi,\eta})\alpha_N(x)$ a left multiplier of $\delta_N^{1/2}$, and the closure of $((\iota \otimes \omega_{\xi,\eta})\alpha_N(x))\delta_N^{1/2}$ equals $(\iota \otimes \omega_{\delta_M^{-1/2}\xi,\eta})\alpha_N(x\delta^{1/2})$. By the concrete formula for U in Definition-Proposition 6.3.11, we conclude that this last operator is in $\mathscr{N}_{\varphi_N}$, and that its image under Λ_N equals $(\iota \otimes \omega_{\xi,\eta})(U)\Gamma_N(x)$. Then by the closedness of Γ_N, we can conclude that for x of the above form, $(\iota \otimes \omega)(\alpha_N(x)) \in \mathscr{N}_{\psi_N}$ for every $\omega \in M_*$, with

$$\Gamma_N((\iota \otimes \omega)(\alpha_N(x))) = (\iota \otimes \omega)(U)\Gamma_N(x).$$

Since such x form a σ-strong-norm core for Γ_N, the same statement holds for a general $x \in \mathscr{N}_{\psi_N}$. From this, it is standard to conclude the invariance: take $\omega = \omega_{\xi,\xi} \in M_*^+$ and $x = y^*y \in \mathscr{M}_{\psi_N}^+$. Let ξ_i denote an orthonormal basis for $\mathscr{L}^2(M)$. Then by the lower-semi-continuity of ψ_N, we find

$$\begin{aligned}
&\psi_N((\iota \otimes \omega_{\xi,\xi})(\alpha_N(y^*y))) \\
&= \psi_N(\sum_n (\iota \otimes \omega_{\xi,\xi_n})(\alpha_N(y))^*((\iota \otimes \omega_{\xi,\xi_n})(\alpha_N(y)))) \\
&= \sum_n \psi_N((\iota \otimes \omega_{\xi,\xi_n})(\alpha_N(y))^*((\iota \otimes \omega_{\xi,\xi_n})(\alpha_N(y))) \\
&= \sum_n \|\Gamma_N((\iota \otimes \omega_{\xi,\xi_n})(\alpha_N(y)))\|^2 \\
&= \sum_n \|(\iota \otimes \omega_{\xi,\xi_n})(U)\Gamma_N(y))\|^2 \\
&= \langle \Gamma_N(y), (\sum_n ((\iota \otimes \omega_{\xi_n,\xi})(U^*)(\iota \otimes \omega_{\xi,\xi_n})(U))\Gamma_N(y))\rangle \\
&= \langle \Gamma_N(y), (\iota \otimes \omega_{\xi,\xi})(U^*U)\Gamma_N(y)\rangle \\
&= \psi_N(y^*y)\omega_{\xi,\xi}(1),
\end{aligned}$$

hence $\psi_N((\iota \otimes \omega)(\alpha_N(x))) = \psi_N(x)\omega(1)$.

□

Remark: It is natural to ask if there is a corresponding result for a general Galois coaction α. We briefly show that one can not expect *too* much: there does not have to exist an invariant nsf operator valued weight T'_α, i.e. an operator valued weight $N^+ \to (N^\alpha)^{+,\mathrm{ext}}$ such that $T'_\alpha((\iota\otimes\omega)\alpha(x)) = \omega(1)T'_\alpha(x)$ for $\omega \in M_*^+$ and $x \in \mathscr{M}^+_{T'_\alpha}$. To give an explicit example, suppose α is an outer left coaction of a von Neumann algebraic quantum group M on a factor N. Then by outerness, there is a *unique* nsf operator valued weight $(M \ltimes N)^+ \to \alpha(N)^{+,\mathrm{ext}}$ (up to a scalar), namely $(\iota \otimes \varphi_{\widehat{M}})\widehat{\alpha}$, where $\widehat{\alpha}$ is the dual right coaction. But if $\widehat{M}$ is not unimodular, then this operator valued weight is not invariant. On the other hand, this does *not* rule out the possibility that there exists an invariant nsf *weight*: for if the original coaction has an invariant nsf weight ψ_N (for example, the coactions occurring in [86]), then one checks that $x \in (M \ltimes N)^+ \to \psi_{\widehat{M}}((\iota_M \otimes \psi_N)(x)) \in [0, +\infty]$ is a well-defined $\widehat{\alpha}$-invariant nsf weight on $M \ltimes N$. We do not know of any example of a Galois coaction without invariant weights.

Lemma 7.2.13. *The one-parametergroups P_N^{it} and $J_N\delta_N^{it}J_N$ commute.*

Proof. Choose x in the Tomita algebra of φ_N. Since P_N^{it}, by its definition, commutes with each ∇_N^{is}, we have that τ_t^N induces automorphisms of the Tomita algebra of φ_N, hence

$$\begin{aligned} P_N^{it}J_N\Lambda_N(x) &= P_N^{it}\Lambda_N(\sigma^N_{i/2}(x)^*) \\ &= \nu_M^{t/2}\Lambda_N(\sigma^N_{i/2}(\tau_t^N(x))^*) \\ &= J_N P_N^{it}\Lambda_N(x), \end{aligned}$$

and P_N^{it} commutes with J_N.

Further, since $\tau_t^N(\delta_N^{is}) = \delta_N^{is}$, we also have that P_N^{it} commutes with δ_N^{is}, and the lemma follows.

□

By the previous lemma, we can define a new one-parametergroup of unitaries $\nabla_{\widehat{N}}^{it} = P_N^{it}J_N\delta_N^{it}J_N$.

Proposition 7.2.14. *Let N be a right Galois object for a von Neumann algebraic quantum group M. Then $\nabla_{\widehat{N}}^{-it}\widehat{\pi}'_{\alpha_N}(m)\nabla_{\widehat{N}}^{it} = \widehat{\pi}'_{\alpha_N}(\sigma_t^{\widehat{M}'}(m))$ for $m \in \widehat{M}'$.*

Proof. By an easy adjustment of Lemma 6.4.15, and using the relative invariance property of δ_N^{it}, we get that $\nabla_{\hat{N}}^{it}\Lambda_{\psi_N}(x) = \Lambda_{\psi_N}(\tau_t^N(x)\delta_N^{-it})$ for $x \in \mathscr{N}_{\psi_N}$, If we apply $(\iota \otimes \omega)(U)$ to this with $\omega \in M_*$, then, using the commutation rules between α_N, τ_t^N and δ_N^{it}, we get

$$(\iota \otimes \omega)(U)\nabla_{\hat{N}}^{it}\Lambda_{\psi_N}(x) = \Lambda_{\psi_N}(\tau_t^N((\iota \otimes \omega(\tau_t^M(\,\cdot\,)\delta_M^{-it}))\alpha_N(x))\delta_N^{-it}).$$

This shows

$$\nabla_{\hat{N}}^{-it}\hat{\pi}'_{\alpha_N}((\iota \otimes \omega)(V_M))\nabla_{\hat{N}}^{it} = \hat{\pi}'_{\alpha_N}((\iota \otimes \omega(\tau_t^M(\,\cdot\,)\delta_M^{-it}))(V_M)).$$

But by doing this same calculation with $N = M$, and using that in this case P_M, δ_M and $\nabla_{\widehat{M}}$, as constructed for the Galois object (M, Δ_M), coincide with the original operators, by the known commutation relations for von Neumann algebraic quantum groups, we get that

$$\nabla_{\widehat{M}}^{-it}((\iota \otimes \omega)(V_M))\nabla_{\widehat{M}}^{it} = (\iota \otimes \omega(\tau_t^M(\,\cdot\,)\delta_M^{-it}))(V_M).$$

Since $\nabla_{\widehat{M}}^{it} = \nabla_{\widehat{M}'}^{-it}$, we get that

$$\nabla_{\hat{N}}^{-it}\hat{\pi}'_{\alpha_N}((\iota \otimes \omega)(V_M))\nabla_{\hat{N}}^{it} = \hat{\pi}'_{\alpha_N}(\sigma_t^{\widehat{M}'}((\iota \otimes \omega)(V_M))).$$

Then of course the same holds with $(\iota\otimes\omega)(V_M)$ replaced by a general element of $\widehat{M}'$, thus proving the proposition.

□

Proposition 7.2.15. *The following commutation relations hold:*

1. $(\nabla_M^{it} \otimes \nabla_{\hat{N}}^{it})\tilde{G} = \tilde{G}(\nabla_N^{it} \otimes \nabla_{\hat{N}}^{it})$,
2. $(\nabla_{\widehat{M}}^{it} \otimes P_N^{it})\tilde{G} = \tilde{G}(\nabla_{\hat{N}}^{it} \otimes P_N^{it}\delta_N^{it})$.

Proof. The first formula follows by the second formula in Proposition 7.2.7, and the fact that the second leg of $\tilde{G}$ lies in N. The second formula follows from the fact that also $\nabla_{\widehat{M}}^{it} = J_M\delta_M^{it}J_M P_M^{it}$, then using the third formula of Proposition 7.2.7 and the first formula in Lemma 7.2.11 together with the definition of δ_N.

□

Proposition 7.2.16. *Up to a positive constant, ψ_N is the only invariant, and φ_N the only δ_M-invariant weight on N.*

Proof. The claim about φ_N follows immediately by Lemma 3.9 of [85] and the fact that α_N is ergodic. The second statement can be proven in the same fashion.

□

We remark that of course all results hold as well in the context of *left Galois coactions*: if (N, γ_N) is a left Galois object for a von Neumann algebraic quantum group P, then $(N, \gamma_N^{\mathrm{op}})$ is a right Galois object for P^{cop}, and in this way, we can apply the constructions of this section to left Galois objects. In particular, with $\psi_N = (\psi_P \otimes \iota)\gamma_N$, by the Galois unitary for the left Galois object (N, γ_N) we shall mean the unitary

$$\tilde{H} : \mathscr{L}^2(N) \otimes \mathscr{L}^2(N) \to \mathscr{L}^2(N) \otimes \mathscr{L}^2(P)$$

$$\Lambda_{\psi_N}(x) \otimes \Lambda_{\psi_N}(y) \to (\Lambda_{\psi_P} \otimes \Lambda_{\psi_N})(\gamma_N(x)(1 \otimes y)), \quad x, y \in \mathscr{N}_{\psi_N}.$$

We end this section by showing that a right *-Galois object for a *-algebraic quantum group (see Definition 3.9.1) can be completed to a right Galois object for the associated von Neumann algebraic quantum group. The fact that a *-algebraic quantum group can be completed to a von Neumann algebraic quantum group (or rather a C*-algebraic quantum group) was shown in [53].

Proposition 7.2.17. *Let A be a *-algebraic quantum group, and M its associated von Neumann algebraic quantum group. Let (B, α_B) be a right *-Galois object for A. Then one can construct canonically a right M-Galois object N, whose underlying von Neumann algebra contains B as a σ-weak dense sub-*-algebra, and such that $\alpha_N(b)(1 \otimes a) = \alpha_B(b)(1 \otimes a)$ for $b \in B$ and $a \in A$.*

Proof. By the discussion in Section 3.9, we conclude that B, endowed with the scalar product

$$\langle b', b \rangle = \varphi_B(b^* \cdot b),$$

is a pre-Hilbert space. Let $\mathscr{L}^2(B)$ be its completion. We will denote the image of b in $\mathscr{L}^2(B)$ by $\Lambda_B(b)$.

Now define

$$\tilde{G}_B : \Lambda_B(B) \odot \Lambda_B(B) \to \Lambda_M(A) \odot \Lambda_B(B) :$$

$$\Lambda_B(b) \otimes \Lambda_B(b') \to \Lambda_M(b_{(1)}) \otimes \Lambda_B(b_{(0)}b').$$

This is easily checked to be a surjective isometry, hence it extends to a unitary

$$\mathscr{L}^2(B) \otimes \mathscr{L}^2(B) \to \mathscr{L}^2(M) \otimes \mathscr{L}^2(B),$$

which we will denote by the same symbol. Clearly, if $b, b' \in B$ and $a \in A$, we have

$$(\omega_{\Lambda_B(b),\Lambda_M(a)} \otimes \iota)(\tilde{G}_B)\Lambda_B(b') = \Lambda_B(\varphi_A(a^* b_{(1)})\, b_{(0)} b').$$

Since any element of B can be written as a linear combination of elements of the form $\varphi_A(a^* b_{(1)})\, b_{(0)}$, we conclude that the operators

$$\pi_B(b) : \Lambda_B(B) \to \Lambda_B(B) : \Lambda_B(b') \to \Lambda_B(bb')$$

extend to bounded operators on $\mathscr{L}^2(B)$, and then π clearly becomes a faithful *-homomorphism $B \to B(\mathscr{L}^2(B))$. We will from now on identify B with its image $\pi_B(B)$.

Let N be the σ-weak closure of B, which is then a von Neumann algebra containing $1_{B(\mathscr{L}^2(B))}$. Since

$$\tilde{G}_B(b \otimes 1)\tilde{G}_B^*(a \otimes 1) = \alpha_B^{\mathrm{op}}(b)(a \otimes 1),$$

which lies in $A \odot B$, we must have that $\tilde{G}_B(N \otimes 1)\tilde{G}_B^* \subseteq N \otimes M$. Denote

$$\alpha_N : N \to N \otimes M : x \to \Sigma\tilde{G}_B(x \otimes 1)\tilde{G}_B^*\Sigma.$$

Then α_N is a normal unital faithful *-homomorphism. Moreover, for $b, b' \in B$ and $a \in A$, we have $\alpha_B(b)(1 \otimes a) = \alpha_N(b)(1 \otimes a)$ and $(b' \otimes 1)\alpha_B(b) = (b' \otimes 1)\alpha_N(b)$, and hence

$$(b' \otimes 1 \otimes 1)((\alpha_N \otimes \iota_M)\alpha_N(b))(1 \otimes 1 \otimes a) = (b' \otimes 1 \otimes 1)((\iota_N \otimes \Delta_M)\alpha_N(b))(1 \otimes 1 \otimes a),$$

from which we conclude that α_N is a coaction.

We have to show now that (N, α_N) is a right M-Galois object. First remark that by Theorem 3.9.4, and the fact that σ_B has positive eigenvalues, we can extend σ_B to a complex one-parametergroup σ_z^B on B such that $\sigma_{-i}^B = \sigma_B$ (see the end of section 4.4). If we then define $U(z)\Lambda_B(b) := \Lambda_B(\sigma_z^B(b))$, then we clearly have the structure of *a* Tomita algebra. Hence there exists a unique nsf weight φ_N on N, such that $\varphi_N(b) = \varphi_B(b)$ for $b \in B$, and moreover, we can identify $\mathscr{L}^2(N)$ with $\mathscr{L}^2(B)$. We again denote the GNS map for φ_N by Λ_N, and we identify Λ_B with Λ_N restricted to B.

Now let $b \in B$. Then for $b' \in B$, we compute:

$$\begin{aligned}
\omega_{\Lambda_N(b'),\Lambda_N(b')}(T_{\alpha_N}(b^*b)) &= \varphi_M((\omega_{\Lambda_N(b'),\Lambda_N(b')}\otimes\iota)\alpha_N(b^*b)) \\
&= \varphi_M((\varphi_B\otimes\iota)((b'^*\otimes 1)\alpha_B(b^*b)(b'\otimes 1))) \\
&= \varphi_A((\varphi_B\otimes\iota)((b'^*\otimes 1)\alpha_B(b^*b)(b'\otimes 1))) \\
&= \varphi_B(b^*b)\varphi_B(b'^*b').
\end{aligned}$$

By lower-semicontinuity, we conclude that $T_{\alpha_N}(b^*b)$ is bounded, and equal to $\varphi_B(b^*b)$. Hence $b \in \mathscr{N}_{T_{\alpha_N}}$, and so α_N is integrable.

Now note that similarly as for φ_B, we can extend ψ_B to an nsf weight ψ_N on N. Then we can construct a unitary U, uniquely defined by the property that

$$U(\Lambda_{\psi_N}(b)\otimes\Lambda_{\varphi_M}(a)) = (\Lambda_{\psi_B}\otimes\Lambda_{\varphi_M})(\alpha_B(b)(1\otimes a))$$

for all $b \in B$ and $a \in A$. An easy computation shows that

$$U(b\otimes 1)U^*(1\otimes a) = \alpha_B(b)(1\otimes a)$$

for $b \in B$ and $a \in A$. Hence

$$U(x\otimes 1)U^* = \alpha_N(x)$$

for $x \in N$. Further, for $a, a' \in A$, and $\omega = \omega_{\Lambda_M(a),\Lambda_M(a')}$, we have

$$(\iota\otimes\omega)(U)\Lambda_B(b) = \Lambda_B((\iota\otimes\varphi_A)((1\otimes a'^*)\alpha_B(b)(1\otimes a))).$$

Using the modular property and the fact that $A^2 = A$, we see that for all $\omega \in \widehat{A}$, the left module action of $\widehat{A}$ extends to a homomorphism $\pi_\alpha : \widehat{A} \to B(\mathscr{L}^2(B))$. Now if $b \in B$, then b is in the Tomita algebra of φ_N. If then $a \in A$, and $\omega = \varphi_A(a\,\cdot)$, we get, for $b' \in B$, that

$$\begin{aligned}
\theta_N(b)\pi_\alpha(\omega)\Lambda_N(b') &= \Lambda_N(\omega(b'_{(1)})b'_{(0)}\sigma^B_{-i/2}(b)) \\
&= \Lambda_N(\varphi_A(ab'_{(1)})b'_{(0)}\sigma^B_{-i/2}(b)) \\
&= \Lambda_N(\varphi_B(a^{[2]}b')a^{[1]}\sigma^B_{-i/2}(b)),
\end{aligned}$$

by Proposition 3.4.1. By the Corollary 3.5.2, we conclude that $\theta_N(B)\pi_\alpha(\widehat{A})$ consists of all finite rank operators of the form $\sum_i l_{\Lambda_N(b_i)}l^*_{\Lambda_N(b'_i)}$, where $b_i, b'_i \in B$. In particular, $\theta_N(B)\pi_\alpha(\widehat{A})$ is σ-weakly dense in $B(\mathscr{L}^2(N))$. Now if $x \in N^{\alpha_N}$, it clearly commutes with $\theta_N(B)$. Since x is a coinvariant, it commutes with the first leg of U, and hence it also commutes with all elements of $\pi_\alpha(\widehat{A})$. So then x must be a scalar multiple of the identity, and we conclude that α_N is ergodic.

Since the Galois isometry for α_N clearly coincides with $\tilde{G}_B$, we get that N is a right M-Galois object.

□

7.3 The reflection technique

In this section, we construct a (possibly) new von Neumann algebraic quantum group, starting from a right Galois object. The main technicality consists in constructing its invariant weights.

For the rest of this section, let N be a fixed right Galois object for some von Neumann algebraic quantum group M. We use notation as in the previous section. In particular, denote as before by $\widehat{Q} = \begin{pmatrix} \widehat{Q}_{11} & \widehat{Q}_{12} \\ \widehat{Q}_{21} & \widehat{Q}_{22} \end{pmatrix} = \begin{pmatrix} \widehat{P} & \widehat{N} \\ \widehat{O} & \widehat{M} \end{pmatrix}$ the linking von Neumann algebra between the right $\widehat{M}$-modules $\mathscr{L}^2(M)$ and $\mathscr{L}^2(N)$. We will denote, as already indicated in the part on linking von Neumann algebras (see section 5.5), the natural inclusion $\widehat{Q} \subseteq B\begin{pmatrix} \mathscr{L}^2(N) \\ \mathscr{L}^2(M) \end{pmatrix}$ by $\pi^{\widehat{Q},2}$, and its parts as $\pi_{ij}^{\widehat{Q},2}$, although we will also use the notation $\widehat{\pi}_{ij}^2$, or no symbol at all. We will also identify the $\widehat{Q}_{ij}$ with their parts in $\widehat{Q}$ again.

Lemma 7.3.1. *We have* $\tilde{G}^*(1 \otimes \widehat{N})W_{\widehat{M}} \subseteq \widehat{N} \otimes \widehat{N}$.

Proof. Let x be an element of $\widehat{N}$. As the first leg of $W_{\widehat{M}}$ lies in $\widehat{M}$, and the first leg of $\tilde{G}$ is a left $\widehat{M}'$-module intertwiner, it is clear that for any $m \in \widehat{M}$, we have

$$\tilde{G}^*(1 \otimes x)W_{\widehat{M}}(\theta_{\widehat{M}}(m) \otimes 1) = (\widehat{\theta}_{\alpha_N}(m) \otimes 1)\tilde{G}^*(1 \otimes x)W_{\widehat{M}}.$$

On the other hand, we have to prove that for all $m \in \widehat{M}$,

$$\tilde{G}^*(1 \otimes x)W_{\widehat{M}}(1 \otimes \theta_{\widehat{M}}(m)) = (1 \otimes \widehat{\theta}_{\alpha_N}(m))\tilde{G}^*(1 \otimes x)W_{\widehat{M}}. \qquad (7.1)$$

Now as $\tilde{G}$ is a right $N \rtimes M$-map, we have

$$(1 \otimes \hat{\theta}_{\alpha_N}(m))\tilde{G}^* = \tilde{G}^*(R_{\widehat{M}} \otimes \hat{\theta}_{\alpha_N})(\Delta_{\widehat{M}}(m)),$$

using the fourth commutation relation of Lemma 6.4.10 in a slightly adapted form. Since also

$$W_{\widehat{M}}(1 \otimes \theta_{\widehat{M}}(m)) = (R_{\widehat{M}} \otimes \theta_{\widehat{M}})(\Delta_{\widehat{M}}(m))W_{\widehat{M}},$$

the stated commutation follows from the intertwining property of x, as $x\theta_{\widehat{M}}(m) = \hat{\theta}_{\alpha_N}(m)x$.

□

Denote the corresponding map by

$$\Delta_{\hat{N}} : \hat{N} \to \hat{N} \otimes \hat{N} : x \to \tilde{G}^*(1 \otimes x)W_{\widehat{M}}$$

Then we can also define

$$\Delta_{\hat{O}} : \hat{O} \to \hat{O} \otimes \hat{O} : x \to \Delta_{\hat{N}}(x^*)^*,$$

and

$$\Delta_{\hat{P}} : \hat{P} \to \hat{P} \otimes \hat{P} : x \to \tilde{G}^*(1 \otimes x)\tilde{G},$$

since $\hat{Q}_{21} = (\hat{Q}_{12})^*$ and the span of $\hat{Q}_{12}\hat{Q}_{21}$ is σ-weakly dense in $\hat{P}$. Finally, we denote by $\Delta_{\hat{Q}}$ the map

$$\hat{Q} \to \hat{Q} \otimes \hat{Q} : x_{ij} \to \hat{\Delta}_{ij}(x_{ij}), \qquad x_{ij} \in \hat{Q}_{ij},$$

where we denote $\hat{\Delta}_{11} = \Delta_{\hat{P}}$, $\hat{\Delta}_{12} = \Delta_{\hat{N}}$, $\hat{\Delta}_{21} = \Delta_{\hat{O}}$ and $\hat{\Delta}_{22} = \Delta_{\widehat{M}}$. Then $\Delta_{\hat{Q}}$ is easily seen to be a (non-unital) normal *-homomorphism.

Lemma 7.3.2. *The map $\Delta_{\hat{Q}}$ is coassociative.*

Proof. This follows trivially by Proposition 7.2.5. □

Since $J_N\hat{\pi}'_{\alpha_N}(m)^*J_N = \hat{\pi}'_{\alpha_N}(J_M m^* J_M)$ for $m \in \widehat{M}'$, we can define a unital anti-*-automorphism $R_{\hat{Q}} : \hat{Q} \to \hat{Q}$ by sending $x \in \hat{Q}_{12}$ to $J_M x^* J_N \in \hat{Q}_{21}$, and then extending it in the natural way.

Lemma 7.3.3. *We have $\Delta_{\hat{Q}}(R_{\hat{Q}}(x)) = (R_{\hat{Q}} \otimes R_{\hat{Q}})\Delta_{\hat{Q}}^{op}(x)$ for $x \in \hat{Q}$.*

Proof. We only have to check whether

$$\tilde{G}^*(1 \otimes J_N x J_M) W_{\widehat{M}} = (J_N \otimes J_N)\Sigma \tilde{G}^*(1 \otimes x) W_{\widehat{M}} \Sigma (J_M \otimes J_M)$$

for $x \in \widehat{Q}_{12}$. But using Lemma 7.2.4 twice, once for N and once for M itself, the right hand side reduces:

$$\begin{aligned} &(J_N \otimes J_N)\Sigma \tilde{G}^*(1 \otimes x) W_{\widehat{M}} \Sigma (J_M \otimes J_M) \\ &= \tilde{G}^*(J_{\widehat{M}} \otimes J_N)\Sigma U^* \Sigma (1 \otimes x) W_{\widehat{M}} \Sigma (J_M \otimes J_M) \\ &= \tilde{G}^*(J_{\widehat{M}} \otimes J_N)(1 \otimes x)\Sigma V_M^* \Sigma W_{\widehat{M}} \Sigma (J_M \otimes J_M) \\ &= \tilde{G}^*(1 \otimes J_N x J_M) W_{\widehat{M}}. \end{aligned}$$

□

In particular, this provides $\widehat{P}$ with the structure of a coinvolutive Hopf-von Neumann algebra structure. Our next goal is to find a left invariant nsf weight for it.

We have shown in Proposition 7.2.14 that the modular automorphism group of $\varphi_{\widehat{M}'}$ on $\widehat{M}'$ can be implemented on $\mathscr{L}^2(N)$ by the one-parametergroup $\nabla_{\widehat{N}}^{it} = P_N^{it} J_N \delta_N^{it} J_N$. Then by Proposition 5.5.5, we can construct an nsf weight $\varphi_{\widehat{P}}$ on $\widehat{P}$ which has $\nabla_{\widehat{N}}$ as spatial derivative with respect to $\varphi_{\widehat{M}'} = \varphi'_{\widehat{M}}$. Then we can also consider the balanced weight $\varphi_{\widehat{Q}} = \varphi_{\widehat{P}} \oplus \varphi_{\widehat{M}}$ on $\widehat{Q}$. Its modular automorphism group $\sigma_t^{\varphi_{\widehat{Q}}}$, which we will denote by $\sigma_t^{\widehat{Q}}$, is then implemented by $\nabla_{\widehat{N}}^{it} \oplus \nabla_{\widehat{M}}^{it}$ if we use the faithful representation $\pi^{\widehat{Q},2}$ of $\widehat{Q}$ on $\mathscr{L}^2(N) \oplus \mathscr{L}^2(M)$.

We make the identification

$$(\mathscr{L}^2(\widehat{Q}), \pi_{\widehat{Q}}, \Lambda_{\varphi_{\widehat{Q}}}) \cong (\begin{pmatrix} \frac{\mathscr{L}^2(\widehat{P})}{\mathscr{L}^2(N)} & \mathscr{L}^2(N) \\ & \mathscr{L}^2(M) \end{pmatrix}, \pi_{\widehat{Q}}, (\widehat{\Lambda}_{ij}))$$

of the natural semi-cyclic representations of $\widehat{Q}$ w.r.t. $\varphi_{\widehat{Q}}$, as explained in section 5.5 (using the obvious notation-wise adaptation w.r.t $\widehat{\ }$ on the right side). We then also write $\widehat{\Lambda}_{ij} = \Lambda_{\widehat{N}}$ for example, and we will also write $\Lambda_{\widehat{M}} \otimes \Lambda_{\widehat{N}}$ for the restriction of the GNS-map of $\varphi_{\widehat{Q}} \otimes \varphi_{\widehat{Q}}$ to $\widehat{M} \otimes \widehat{N} \subseteq \mathscr{N}_{\varphi_{\widehat{Q}} \otimes \varphi_{\widehat{Q}}}$.

We will now provide another formula for $\tilde{G}^*$.

Proposition 7.3.4. *Let N be a right Galois object for a von Neumann algebraic quantum group M. If $m \in \mathscr{N}_{\varphi_{\widehat{M}}}$ and $x \in \hat{N} \cap \mathscr{N}_{\varphi_{\hat{Q}}}$, then $\Delta_{\hat{N}}(x)(m\otimes 1) \in \mathscr{D}(\Lambda_{\hat{N}} \otimes \Lambda_{\hat{N}})$ and*

$$(\Lambda_{\hat{N}} \otimes \Lambda_{\hat{N}})(\Delta_{\hat{N}}(x)(m \otimes 1)) = \tilde{G}^*(\Lambda_{\widehat{M}}(m) \otimes \Lambda_{\hat{N}}(x)).$$

Proof. Since

$$\begin{aligned}&(\iota \otimes \varphi_{\hat{Q}})((m^* \otimes 1)\widehat{\Delta}_{12}(x)^*\widehat{\Delta}_{12}(x)(m \otimes 1))\\ &\quad= (\iota \otimes \varphi_{\widehat{M}})((m^* \otimes 1)\Delta_{\widehat{M}}(x^*x)(m \otimes 1))\\ &\quad= \varphi_{\hat{Q}}(x^*x)m^*m\end{aligned}$$

for $x \in \widehat{Q}_{12}$ and $m \in \widehat{M}$, it is clear that $\widehat{\Delta}_{12}(x)(m \otimes 1) \in \mathscr{D}(\hat{\Lambda}_{12} \otimes \hat{\Lambda}_{12})$ for $m \in \mathscr{N}_{\varphi_{\widehat{M}}}$ and $x \in \widehat{Q}_{12} \cap \mathscr{N}_{\varphi_{\hat{Q}}}$, and that the map

$$\widehat{\Lambda}_{22}(m) \otimes \widehat{\Lambda}_{12}(x) \to (\hat{\Lambda}_{12} \otimes \hat{\Lambda}_{12})(\hat{\Delta}_{12}(x)(m \otimes 1))$$

extends to a well-defined isometry. We now show that it coincides with $\tilde{G}^*$.

Let z be an element of $\mathscr{N}_{\varphi_{\widehat{M}'}}$. Then it is sufficient to prove that

$$\widehat{\Delta}_{12}(x)(\Lambda_{\widehat{M}}(m) \otimes \Lambda_{\widehat{M}'}(z)) = (1 \otimes \widehat{\pi}'_{\alpha_N}(z))\tilde{G}^*(\widehat{\Lambda}_{22}(m) \otimes \widehat{\Lambda}_{12}(x)).$$

But $\widehat{\Delta}_{12}(x) = \tilde{G}^*(1 \otimes x)W_{\widehat{M}}$, and bringing $\tilde{G}$ to the other side, $\tilde{G}(1 \otimes \widehat{\pi}'_{\alpha_N}(z))\tilde{G}^*$ can be written as $\Sigma U(1 \otimes J_{\widehat{M}}R_{\widehat{M}'}(z)^*J_{\widehat{M}})U^*\Sigma$ by the remarks in the proof of Lemma 6.4.10. Taking a scalar product in the first factor, it is then sufficient to prove that for $\omega \in (\widehat{M'})_*$, we have

$$x(\omega \otimes \iota)(W_{\widehat{M}})\Lambda_{\widehat{M}'}(z) = (\iota \otimes \omega)(U(1 \otimes J_{\widehat{M}}R_{\widehat{M}'}(z)J_{\widehat{M}})U^*)\widehat{\Lambda}_{12}(x).$$

But now using again that $(\widehat{\pi}'_{\alpha_N} \otimes \iota)(V_M) = U$, it is sufficient to show that

$$(\iota \otimes \omega)(V_M(1 \otimes J_{\widehat{M}}R_{\widehat{M}'}(z)J_{\widehat{M}})V_M^*) \in \mathscr{N}_{\varphi_{\widehat{M}'}}$$

and that applying $\Lambda_{\widehat{M}'}$ to it gives $(\omega \otimes \iota)(W_{\widehat{M}})\Lambda_{\widehat{M}'}(z)$. We could check this directly, but we can just as easily backtrack our arguments: we only have to see if for $y \in \mathscr{N}_{\varphi_{\widehat{M}}}$, we have

$$y(\omega \otimes \iota)(W_{\widehat{M}})\Lambda_{\widehat{M}'}(z) = (\iota \otimes \omega)(V_M(1 \otimes J_{\widehat{M}}R_{\widehat{M}'}(z)J_{\widehat{M}})V_M^*)\Lambda_{\widehat{M}}(y)$$

for any $z \in \mathscr{N}_{\varphi_{\widehat{M}'}}$. This is then seen to be the same as saying that

$$(\Lambda_{\widehat{M}} \otimes \Lambda_{\widehat{M}})(\Delta_{\widehat{M}}(y)(m \otimes 1)) = W^*_{\widehat{M}}(\Lambda_{\widehat{M}}(m) \otimes \Lambda_{\widehat{M}}(y)),$$

which is of course true by definition.

□

Lemma 7.3.5. *Let x be in $\mathscr{N}_{\varphi_N} \cap \mathscr{N}_{\varphi_N}^*$, and $a \in \mathscr{T}_{\varphi_M}$, the Tomita algebra for φ_M. Then*

$$(\omega_{\Lambda_N(x^*),\Lambda_M(\sigma_i^M(a)^*)} \otimes \iota)(\tilde{G}) = (\omega_{\Lambda_M(a),\Lambda_N(x)} \otimes \iota)(\tilde{G}^*).$$

Proof. Choose $\omega \in N_*$. Then

$$\begin{aligned} \omega((\omega_{\Lambda_N(x^*),\Lambda_M(\sigma_i^M(a)^*)} \otimes \iota)(\tilde{G})) &= \varphi_M(\sigma_i^M(a)((\omega \otimes \iota)(\alpha_N(x)^*))) \\ &= \varphi_M(((\omega \otimes \iota)(\alpha_N(x)^*))a) \\ &= \langle \Lambda_M(a), \Lambda_M((\overline{\omega} \otimes \iota)\alpha_N(x))\rangle \\ &= \langle \Lambda_M(a), (\iota \otimes \overline{\omega})(\tilde{G})\Lambda_N(x)\rangle \\ &= \omega((\omega_{\Lambda_M(a),\Lambda_N(x)} \otimes \iota)(\tilde{G}^*)) \end{aligned}$$

□

Proposition 7.3.6. *Let (N, α_N) be a right Galois object for a von Neumann algebraic quantum group M. If $x \in \hat{N} \cap \mathscr{N}_{\varphi_{\hat{Q}}}$ and $y \in \hat{O} \cap \mathscr{N}_{\varphi_{\hat{Q}}}$, then $\Delta_{\hat{O}}(y)(x \otimes 1)$ in $\mathscr{D}(\Lambda_{\widehat{M}} \otimes \Lambda_{\hat{O}})$, and*

$$(\Lambda_{\widehat{M}} \otimes \Lambda_{\hat{O}})(\Delta_{\hat{O}}(y)(x \otimes 1)) = (J_M \otimes J_{\hat{N}})\tilde{G}(J_N \otimes J_{\hat{O}})(\Lambda_{\hat{N}}(x) \otimes \Lambda_{\hat{O}}(y)).$$

Remark: Compare this formula with the identity $(J_{\widehat{M}} \otimes J_M)W_M(J_{\widehat{M}} \otimes J_M) = W_M^*$.

Proof. It is sufficient to prove that for y in $\hat{Q}_{21} \cap \mathscr{N}_{\varphi_{\hat{Q}}}$ and $\omega \in \hat{Q}_*$, we have $(\omega \otimes \iota)(\hat{\Delta}_{21}(y)) \in \mathscr{N}_{\varphi_{\hat{Q}}}$, and

$$\hat{\Lambda}_{21}((\omega \otimes \iota)(\hat{\Delta}_{21}(y))) = (\omega \otimes \iota)((J_M \otimes J_{\hat{N}})\tilde{G}(J_N \otimes J_{\hat{O}}))\hat{\Lambda}_{21}(y).$$

Indeed: supposing this holds, choose $z \in \mathscr{N}_{\varphi'_{\hat{Q}}}$. Then

$$((\omega \otimes \iota)(\hat{\Delta}_{21}(y)))\Lambda_{\varphi'_{\hat{Q}}}(z) = \pi_{\hat{Q}'}(z)(\omega \otimes \iota)((J_M \otimes J_{\hat{N}})\tilde{G}(J_N \otimes J_{\hat{O}}))\Lambda_{\varphi_{\hat{Q}}}(y).$$

Choosing $x \in \mathscr{N}_{\varphi_{\hat{Q}}} \cap \hat{N}$, this implies

$$\begin{aligned} &\hat{\Delta}_{21}(y)(\Lambda_{\varphi_{\hat{Q}}}(x) \otimes \Lambda_{\varphi'_{\hat{Q}}}(z)) \\ &= (1 \otimes \pi_{\hat{Q}'}(z))(J_M \otimes J_{\hat{N}})\tilde{G}(J_N \otimes J_{\hat{O}})(\Lambda_{\varphi_Q}(x) \otimes \Lambda_{\varphi_{\hat{Q}}}(y)). \end{aligned}$$

Choosing also $w \in \mathscr{N}_{\varphi'_{\hat{Q}}}$, and multiplying the previous expression to the left with $\pi_{\hat{Q}'}(w)$, we obtain

$$\begin{aligned}&\hat{\Delta}_{21}(y)(x \otimes 1)(\Lambda_{\varphi'_{\hat{Q}}}(w) \otimes \Lambda_{\varphi'_{\hat{Q}}}(z))\\ &\quad = \quad (\pi_{\hat{Q}'}(w) \otimes \pi_{\hat{Q}'}(z))(J_M \otimes J_{\hat{N}})\tilde{G}(J_N \otimes J_{\hat{O}})(\Lambda_{\varphi_Q}(x) \otimes \Lambda_{\varphi_{\hat{Q}}}(y)).\end{aligned}$$

Since $\mathscr{N}_{\varphi'_{\hat{Q}}} \odot \mathscr{N}_{\varphi'_{\hat{Q}}}$ is a core for $\Lambda_{\varphi'_{\hat{Q}} \otimes \varphi'_{\hat{Q}}}$, we obtain $\hat{\Delta}_{21}(y)(x \otimes 1) \in \mathscr{N}_{\varphi_{\hat{Q}} \otimes \varphi_{\hat{Q}}}$ and

$$(\hat{\Lambda}_{22} \otimes \hat{\Lambda}_{21})(\hat{\Delta}_{21}(y)(x \otimes 1)) = (J_M \otimes J_{\hat{N}})\tilde{G}(J_N \otimes J_{\hat{O}})(\hat{\Lambda}_{12}(x) \otimes \hat{\Lambda}_{21}(y))$$

by Proposition 5.3.6.

Now the identity

$$\hat{\Lambda}_{21}((\omega \otimes \iota)(\hat{\Delta}_{21}(y))) = (\omega \otimes \iota)((J_M \otimes J_{\hat{N}})\tilde{G}(J_N \otimes J_{\hat{O}}))\hat{\Lambda}_{21}(y)$$

is equivalent with

$$J_{\hat{O}}\hat{\Lambda}_{21}((\omega \otimes \iota)(\hat{\Delta}_{21}(y))) = (\overline{\omega}(J_M(\,\cdot\,)^* J_N) \otimes \iota)(\tilde{G})J_{\hat{O}}\hat{\Lambda}_{21}(y). \tag{7.2}$$

We will first prove this identity for special elements y and ω.

Let $y \in \hat{Q}_{21} \cap \mathscr{N}_{\varphi_{\hat{Q}}}$ be in the Tomita algebra of $\varphi_{\hat{Q}}$. Let ω be of the form $\omega_{\Lambda_N(x),\Lambda_M(a)}$ with x, a in the Tomita algebra of respectively φ_N and φ_M. Then by the first formula of Lemma 7.2.15 (used both in the general case and the case where $N = M$), we have that $(\omega \otimes \iota)(\hat{\Delta}_{21}(y))$ will also be analytic for $\sigma_t^{\hat{Q}}$, with

$$\sigma_{-i/2}^{\hat{Q}}((\omega_{\Lambda_N(x),\Lambda_M(a)} \otimes \iota)(\hat{\Delta}_{21}(y))$$

equal to

$$(\omega_{\nabla_N^{1/2}\Lambda_N(x),\nabla_M^{-1/2}\Lambda_M(a)} \otimes \iota)(\hat{\Delta}_{21}(\sigma_{-i/2}^{\hat{Q}}(y))).$$

For this, we only have to observe that

$$z \to (\omega_{\nabla_N^{iz}\Lambda_N(x),\nabla_M^{i\overline{z}}\Lambda_M(a)} \otimes \tilde{\omega})(\hat{\Delta}_{21}(\sigma_z^{\hat{Q}}(y)))$$

is an analytic function for any $\tilde{\omega} \in \hat{Q}_*$. Further,

$$(\omega \otimes \iota)(\hat{\Delta}_{21}(y))^* = (\overline{\omega} \otimes \iota)(\hat{\Delta}_{12}(y^*)),$$

which will be in $\mathscr{D}(\widehat{\Lambda}_{12})$ by Proposition 7.3.4, with

$$\widehat{\Lambda}_{12}((\overline{\omega}\otimes\iota)(\widehat{\Delta}_{12}(y^*))) = (\overline{\omega}\otimes\iota)(\tilde{G}^*)\widehat{\Lambda}_{12}(y^*).$$

This shows that $(\omega\otimes\iota)(\widehat{\Delta}_{21}(y)) \in \mathscr{D}(\widehat{\Lambda}_{21})$.

Now by Proposition 7.3.4 and Lemma 7.3.5, we have then also

$$\widehat{\Lambda}_{12}((\overline{\omega}\otimes\iota)(\widehat{\Delta}_{12}(y^*))) = (\omega_{\Lambda_N(x^*),\Lambda_M(\sigma^M_{-i}(a^*))}\otimes\iota)(\tilde{G})\widehat{\Lambda}_{12}(y^*),$$

and by Lemma 7.2.15, we have that $(\omega_{\Lambda_N(x^*),\Lambda_M(\sigma^M_{-i}(a^*))}\otimes\iota)(\tilde{G})$ is analytic for $\chi^N_t = \mathrm{Ad}(\nabla^{it}_{\hat{N}})$, with

$$\chi^N_{-i/2}((\omega_{\Lambda_N(x^*),\Lambda_M(\sigma^M_{-i}(a^*))}\otimes\iota)(\tilde{G})) = (\omega_{J_N\Lambda_N(x),J_M\Lambda_M(a)}\otimes\iota)(\tilde{G}).$$

Now if $w \in B(\mathscr{L}^2(N))$ is analytic for χ^N_t, this means means that $\nabla^{iz}_{\hat{Q}} w \nabla^{-iz}_{\hat{Q}}$ is bounded for any $z \in \mathbb{C}$, its closure being precisely $\chi^N_z(w)$. So combining all this, we get

$$\begin{aligned} & J_{\hat{O}}\widehat{\Lambda}_{21}((\omega\otimes\iota)(\widehat{\Delta}_{21}(y))) \\ &= \nabla^{1/2}_{\hat{Q}}\widehat{\Lambda}_{12}((\omega\otimes\iota)(\widehat{\Delta}_{21}(y))^*) \\ &= (\nabla^{1/2}_{\hat{Q}}(\omega_{\Lambda_N(x^*),\Lambda_M(\sigma^M_{-i}(a^*))}\otimes\iota)(\tilde{G})\nabla^{-1/2}_{\hat{Q}})\nabla^{1/2}_{\hat{Q}}\widehat{\Lambda}_{12}(y^*) \\ &= (\omega_{J_N\Lambda_N(x),J_M\Lambda_M(a)}\otimes\iota)(\tilde{G})J_{\hat{O}}\widehat{\Lambda}_{21}(y) \\ &= (\overline{\omega}(J_M(\,\cdot\,)^*J_N)\otimes\iota)(\tilde{G})J_{\hat{O}}\widehat{\Lambda}_{21}(y). \end{aligned}$$

Now by closedness of $\Lambda_{\hat{Q}}$, this equality remains true for ω arbitrary. Since such y's form a σ-strong-norm core for $\widehat{\Lambda}_{21}$, the equality is true for any $y \in \widehat{Q}_{21} \cap \mathscr{N}_{\varphi_{\hat{Q}}}$.

□

Theorem 7.3.7. *Let (N,α_N) be a right Galois object for a von Neumann algebraic quantum group M. Let $\widehat{P} = \widehat{\theta}_\alpha(\widehat{M})'$ be the coinvolutive Hopf-von Neumann algebra introduced after Lemma 7.3.3. Then the unique nsf weight $\varphi_{\widehat{P}}$ on $\widehat{P}$ satisfying $\frac{d\varphi_{\hat{P}}}{d\varphi'_{\widehat{M}}} = \nabla_{\hat{N}}$ is a left invariant nsf weight on $\widehat{P}$. In particular, $\widehat{P}$ is a von Neumann algebraic quantum group.*

Proof. The previous proposition, together with a small adaptation of Lemma 5.7.8 with regard to the inclusion $N \otimes 1_N \subseteq N \otimes N$ and the operator valued weight $(\iota_N \otimes \varphi_N)$, shows that

$$(\iota \otimes \varphi_{\hat{P}})(\Delta_{\hat{P}}(L_\xi L_\xi^*)) = \varphi_{\hat{P}}(L_\xi L_\xi^*)$$

for ξ right-bounded and in the domain of $\nabla_{\hat{N}}^{1/2}$. From Lemma IX.3.9 of [84], it follows that also $(\iota \otimes \varphi_{\hat{P}})(\Delta_{\hat{P}}(b)) = \varphi_{\hat{P}}(b)$ for $b \in \mathscr{M}_{\varphi_{\hat{P}}}^+$. Indeed: that lemma implies that b can be approximated from below by elements of the form $\sum_{i=1}^n L_{\xi_i} L_{\xi_i}^*$ with ξ_i right-bounded, and since b is integrable, every ξ_i must be in $\mathscr{D}(\nabla_{\hat{N}}^{1/2})$ (cf. Lemma IX.3.12.(i) in [84]). So we can conclude by lower-semi-continuity that $\varphi_{\hat{P}}$ is an nsf left invariant weight. Then if $R_{\hat{P}}$ is a coinvolution for $\hat{P}$, $\psi_{\hat{P}} := \varphi_{\hat{P}} \circ R_{\hat{Q}}$ will be a right invariant nsf weight. Hence $\hat{P}$ is a von Neumann algebraic quantum group.

□

Definition 7.3.8. *If N is a right Galois object for a von Neumann algebraic quantum group M, and $(\hat{P}, \Delta_{\hat{P}})$ the von Neumann algebraic quantum group constructed from it in the foregoing manner, then we call $\hat{P}$ the* reflected von Neumann algebraic quantum group *(or just the reflection) of $\widehat{M}$ across N. We call the dual P of $\hat{P}$ the* reflected von Neumann algebraic quantum group *of M across N.*

7.4 Linking structures

7.4.1 Linking quantum groupoids

The following definition introduces a notion of W*-Morita equivalence which takes a comultiplication structure into account.

Definition 7.4.1. *A* linking weak Hopf-von Neumann algebra *two Hopf-von Neumann algebras*[2] *$\widehat{M}$ and $\hat{P}$ consists of a linking von Neumann algebra $(\hat{Q}, e)$ between $\widehat{M}$ and $\hat{P}$, together with a coassociative normal faithful *-homomorphism $\Delta_{\hat{Q}} : \hat{Q} \to \hat{Q} \otimes \hat{Q}$, whose restriction to $\hat{P}$ and $\widehat{M}$ coincides with respectively $\Delta_{\hat{P}}$ and $\Delta_{\widehat{M}}$. If there exists a linking von Neumann algebraic quantum groupoid between two Hopf-von Neumann algebras $\widehat{M}$ and $\hat{P}$,*

[2] We write them as 'duals' to have compatibility with the previous sections later on.

then we call $\widehat{M}$ *and* $\widehat{P}$ comonoidally W*-Morita equivalent*.*

When $\widehat{M}$ *and* $\widehat{P}$ *are in fact von Neumann algebraic quantum groups, we also call a linking weak Hopf-von Neumann algebra between* $\widehat{M}$ *and* $\widehat{P}$ *a* linking von Neumann algebraic quantum groupoid *between* $\widehat{M}$ *and* $\widehat{P}$*.*

Remark: Note that we do *not* assume that $\Delta_{\widehat{Q}}$ is unital! In fact: by the statement that $\Delta_{\widehat{Q}}$ should restrict to $\Delta_{\widehat{P}}$ and $\Delta_{\widehat{M}}$, we get that

$$\Delta_{\widehat{Q}}(e) = e \otimes e$$

as well as

$$\Delta_{\widehat{Q}}(1_{\widehat{Q}} - e) = (1_{\widehat{Q}} - e) \otimes (1_{\widehat{Q}} - e),$$

so

$$\Delta_{\widehat{Q}}(1_{\widehat{Q}}) = (e \otimes e) + (1_{\widehat{Q}} - e) \otimes (1_{\widehat{Q}} - e),$$

which is *not* the unit in $\widehat{Q} \otimes \widehat{Q}$.

In the following, we will use the notation as for linking von Neumann algebras, but we put an extra ^ on the symbols, and we drop the extra index $\widehat{Q}$ at places.

We refer for example to the discussion concerning linking weak Hopf algebras in subsection 1.2.3 for the intuitive reason for calling this a quantum groupoid.

Just as we can give an abstract notion of a linking algebra without making a reference as to what it is a linking algebra between, one can define the notion of a linking quantum groupoid.

Definition 7.4.2. *A* linking weak Hopf-von Neumann algebra *consists of a triple* $(\widehat{Q}, e, \Delta_{\widehat{Q}})$ *for which* $(\widehat{Q}, e)$ *is a linking von Neumann algebra, and* $\Delta_{\widehat{Q}}$ *is a (non-unital) normal coassociative faithful* **-homomorphism* $\widehat{Q} \rightarrow \widehat{Q} \otimes \widehat{Q}$ *satisfying* $\Delta_{\widehat{Q}}(e) = e \otimes e$ *and* $\Delta_{\widehat{Q}}(1_{\widehat{Q}} - e) = (1_{\widehat{Q}} - e) \otimes (1_{\widehat{Q}} - e)$*.*

A linking von Neumann algebraic quantum groupoid *is a linking weak Hopf-von Neumann algebra whose diagonal corners become von Neumann algebraic quantum groups by restricting the coproduct.*

As in the case of von Neumann algebraic quantum groups, we will always suppose that a linking von Neumann algebraic quantum groupoid comes equipped with *fixed* left invariant nsf weights $\varphi_{\widehat{M}}$ and $\varphi_{\widehat{P}}$ on its corners.

It is clear that a linking von Neumann algebraic quantum groupoid is a linking von Neumann algebraic quantum groupoid between its diagonal corners. But in fact, we do not even have to assume a priori that the underlying couple $(\widehat{Q}, e)$ is a linking von Neumann algebra.

Proposition 7.4.3. *Suppose that in the previous definition for a linking von Neumann algebraic quantum groupoid, we replace $(\widehat{Q}, e)$ by an arbitrary couple consisting of a von Neumann algebra $\widehat{Q}$ and a self-adjoint projection $e \in \widehat{Q}$ which does not lie in the center of $\widehat{Q}$. Then $(\widehat{Q}, e)$ is a linking von Neumann algebra.*

Proof. We have to show that $e_2 := e$ and $e_1 := (1 - e)$ are full. Denote again $\widehat{Q}_{ij} = e_i\widehat{Q}e_j$. Then for $i \neq j$, the σ-weak closure of $\widehat{Q}_{ji}\widehat{Q}_{ij}$ in $\widehat{Q}_{jj}$ is a two-sided ideal, so it must be of the form $p\widehat{Q}_{jj}$ for some projection p in the center of $\widehat{Q}_{jj}$. Since $\widehat{\Delta}_{jj}(\widehat{Q}_{ji}\widehat{Q}_{ij}) \subseteq (\widehat{Q}_{ji}\widehat{Q}_{ij} \otimes \widehat{Q}_{ji}\widehat{Q}_{ij})$, we must have $\widehat{\Delta}_{jj}(p) \leqslant (1 \otimes p)$. Then p must be either 1 or 0 by Lemma 6.4 of [56]. But $\widehat{Q}_{ij}$ is non-zero, by the non-centrality of e. Hence $p = 1$, and the fullness of e and $1_{\widehat{Q}} - e$ follows. □

Remark: The previous proposition easily implies that von Neumann algebraic quantum groupoids correspond exactly to those measured quantum groupoids for which the basis is $\mathbb{C}^2$, and for which the source and target maps coincide and have their image outside the center of the underlying von Neumann algebra. This correspondence will be proven in more detail in Example 11.1.9.

In any case, it is easy to see that the triple $(\widehat{Q}, \begin{pmatrix} 0 & 0 \\ 0 & 1_{\widehat{M}} \end{pmatrix}, \Delta_{\widehat{Q}})$ which we constructed from a right Galois object, will be a linking von Neumann algebraic quantum groupoid. We will show later on that any linking von Neumann algebraic quantum groupoid arises in this way.

Now *fix* a linking von Neumann algebraic quantum groupoid $(\widehat{Q}, e)$ between two von Neumann algebraic quantum groups $\widehat{M}$ and $\widehat{P}$. We will denote $e_1 = (1_{\widehat{Q}} - e)$ and $e_2 = e$. Moreover, we will also write $f_1 = \theta_{\widehat{Q}}(e_1)$ and $f_2 = \theta_{\widehat{Q}}(e_2)$. Finally, since the corners $\widehat{M}$ and $\widehat{P}$ are now in a symmetric position with respect to each other, we will rather suppress the notation $\pi_{\hat{Q}}$ for the standard left representation, and explicitly use the notations $\widehat{\pi}^1$ and $\widehat{\pi}^2$ for the restrictions to the two columns of $\mathscr{L}^2(\hat{Q})$.

By the general theory in the first section of the final chapter, we can define a partial isometry

$$W^*_{\hat{Q}} : \mathscr{L}^2(\hat{Q}) \otimes \mathscr{L}^2(\hat{Q}) \to \mathscr{L}^2(\hat{Q}) \otimes \mathscr{L}^2(\hat{Q}),$$

uniquely determined by

$$W^*_{\hat{Q}}(\Lambda_{\varphi_{\hat{Q}}}(x) \otimes \Lambda_{\varphi_{\hat{Q}}}(y)) = (\Lambda_{\varphi_{\hat{Q}}} \otimes \Lambda_{\varphi_{\hat{Q}}})(\Delta_{\hat{Q}}(y)(x \otimes 1_{\hat{Q}})).$$

Its source projection will be the projection onto the direct sum of the parts $\mathscr{L}^2(\hat{Q}_{kj}) \otimes \mathscr{L}^2(\hat{Q}_{ik})$ of $\mathscr{L}^2(\hat{Q}) \otimes \mathscr{L}^2(\hat{Q})$, with i, j, k ranging over 1 and 2, and its range projection will be the projection onto the direct sum of the parts $\mathscr{L}^2(\hat{Q}_{ij}) \otimes \mathscr{L}^2(\hat{Q}_{ik})$. In fact, $W^*_{\hat{Q}}$ splits into unitaries

$$(\widehat{W}^j_{ik})^* : \mathscr{L}^2(\hat{Q}_{kj}) \otimes \mathscr{L}^2(\hat{Q}_{ik}) \to \mathscr{L}^2(\hat{Q}_{ij}) \otimes \mathscr{L}^2(\hat{Q}_{ik}),$$

determined by the same formula as for $W^*_{\hat{Q}}$.

7.4.2 Co-linking quantum groupoids

We now define abstractly the duals of linking quantum groupoids.

Definition 7.4.4. *A* co-linking von Neumann algebraic quantum groupoid *consists of a von Neumann algebra Q, four non-zero central self-adjoint projections $p_{ij} \in Q$ and a (non-unital) normal coassociative *-homomorphism $\Delta_Q : Q \to Q \otimes Q$, such that $\Delta_Q(p_{ij}) = \sum_{k=1}^2 p_{ik} \otimes p_{kj}$, and such that, denoting $Q_{ij} = p_{ij} \cdot Q$ and*

$$\Delta^k_{ij} : Q_{ij} \to Q_{ik} \otimes Q_{kj} : x \to (p_{ik} \otimes p_{kj})\Delta_Q(x),$$

there exist nsf weights φ^Q_{ij} and ψ^Q_{ij} on Q_{ij} such that

$$(\iota_{Q_{ik}} \otimes \varphi^Q_{kj})(\Delta^k_{ij}(x_{ij})) = \varphi^Q_{ij}(x_{ij}) \cdot 1_{Q_{ik}}$$

for all $x_{ij} \in \mathscr{M}^+_{\varphi^Q_{ij}}$ and

$$(\psi^Q_{ik} \otimes \iota_{Q_{kj}})(\Delta^k_{ij}(x_{ij})) = \psi^Q_{ij}(x_{ij}) \cdot 1_{Q_{kj}}$$

for all $x_{ij} \in \mathscr{M}^+_{\psi^Q_{ij}}$.

Note that in terms of the parts Δ_{ij}^k, the coassociativity condition reads

$$(\Delta_{ik}^l \otimes \iota_{Q_{kj}})\Delta_{ij}^k(x_{ij}) = (\iota_{Q_{il}} \otimes \Delta_{lj}^k)\Delta_{ij}^l(x_{ij})$$

for $x \in Q_{ik}$ and $i, j, k, l \in \{1, 2\}$.

This definition can again be given more succinctly using the language of measured quantum groupoids: co-linking von Neumann algebraic quantum groupoids correspond exactly to those measured quantum groupoids on base space $\mathbb{C}^2$ whose target and source maps *do* end up in the center of the underlying von Neumann algebra, and such that moreover their ranges generate a copy of the algebra $\mathbb{C}^4$. We again make this correspondence exact in Example 11.1.9. We also show there that there is a one-to-one correspondence between linking von Neumann algebraic quantum groupoids and co-linking von Neumann algebraic quantum groupoids, using the 'duality functor' between measured quantum groupoids.

If Q is a linking von Neumann algebraic quantum groupoid, we will also write $Q_{11} = P, Q_{21} = O, Q_{12} = N$ and $Q_{22} = M$. We further personalize the Δ_{ij}^k: we write $\Delta_{11}^1 = \Delta_P, \Delta_{12}^1 = \gamma_N, \Delta_{21}^1 = \alpha_O, \Delta_{22}^1 = \beta_P$ and $\Delta_{22}^2 = \Delta_M, \Delta_{12}^2 = \alpha_N, \Delta_{21}^2 = \gamma_O, \Delta_{22}^1 = \beta_M$. We then also index the weights in the definition by letters instead of numbers when more convenient. We also denote $\varphi_Q = \oplus_{i,j}^2 \varphi_{ij}$, which is now just a direct sum of weights. Note that we can canonically identify $\mathscr{L}^2(Q_{ij})$ with $\pi_Q(e_{ij})\mathscr{L}^2(Q)$, and we will of course do so in the following. Also note that M and P are then von Neumann algebraic quantum groups, with $\varphi_{22} = \varphi_M$, resp. $\varphi_{11} = \varphi_P$ as left invariant nsf weights, so that there is no conflict of notation. It is further easily observed that α_N is a right coaction of M on N, using the (piecewise) coassociativity of Δ_Q (and similarly for the maps α_O, γ_N and γ_O). The maps β_M and β_P will be called the *external comultiplications*.

Just as for linking von Neumann algebraic quantum groupoids, it is easy to show that there is a unique map

$$W_Q : \mathscr{L}^2(Q) \otimes \mathscr{L}^2(Q) \to \mathscr{L}^2(Q) \otimes \mathscr{L}^2(Q)$$

such that for $x, y \in \mathscr{N}_{\varphi_Q}$, we have

$$W_Q^*(\Lambda_{\varphi_Q}(x) \otimes \Lambda_{\varphi_Q}(y)) = (\Lambda_{\varphi_Q} \otimes \Lambda_{\varphi_Q})(\Delta_Q(y)(x \otimes 1)),$$

the right hand side being well-defined. Also, this W_Q again splits up into parts

$$W_{ik}^j : \mathscr{L}^2(Q_{ik}) \otimes \mathscr{L}^2(Q_{kj}) \to \mathscr{L}^2(Q_{ik}) \otimes \mathscr{L}^2(Q_{ij}),$$

with each W_{ik}^j an isometry, and by measured quantum groupoid theory, a unitary. In fact, denoting $(\widehat{Q}, e)$ the dual linking von Neumann algebraic quantum groupoid, we have that

$$W_Q = \Sigma W_{\widehat{Q}}^* \Sigma$$

and

$$W_{ik}^j = \Sigma(\widehat{W}_{ik}^j)^* \Sigma.$$

Now we can also write

$$W_{\widehat{Q}} = \sum_{i,k=1}^{2} \widehat{W}_{ik},$$

where

$$\widehat{W}_{ik} = W_{\widehat{Q}}(1 \otimes e_i f_k) \in \widehat{Q}_{ki} \otimes Q_{ik}$$

for $i, k \in \{1, 2\}$ (with the notation as on page 244). If we then denote by π_{ik} the natural *-representation of Q on $\mathscr{L}^2(\widehat{Q}_{ik})$, we have the following trivial but important lemma:

Lemma 7.4.5. *For all i, k, j, we have $\widehat{W}_{ik}^j = (\widehat{\pi}_{ki}^j \otimes \pi_{ik})(\widehat{W}_{ik})$.*

In the following, we will drop the symbol π_{ik} when we restrict it to Q_{ik}.

Proposition 7.4.6. *Let Q be a co-linking von Neumann algebraic quantum groupoid. Then (N, α_N) is a right Galois object for M.*

Proof. We must show that α_N is integrable and ergodic, and that its Galois isometry $\tilde{G}$ is a unitary.

By one of the invariance formulas in the definition of a co-linking von Neumann algebraic quantum groupoid, we have that

$$(\iota \otimes \varphi_M)\alpha_N(x) = \varphi_N(x) \cdot 1_N$$

for $x \in \mathscr{M}_{\varphi_N}^+$. Clearly this implies that α_N is integrable, since φ_N is semi-finite. Now

$$(\iota \otimes \varphi_M)\alpha_N(x) = \varphi_N(x) \cdot 1_N$$

even holds for $x \in N^+$: this is a consequence of the strong form of left invariance for measured quantum groupoids (see Lemma 11.1.8). This then implies that α_N is ergodic: if we denote again $T_{\alpha_N} = (\iota \otimes \varphi_M)\alpha_N$, then the linear span of $T_{\alpha_N}(\mathscr{M}^+_{T_{\alpha_N}})$ is σ-weakly dense in N^{α_N}. But $\mathscr{M}^+_{T_{\alpha_N}} = \mathscr{M}^+_{\varphi_N}$ by the above equality, hence $N^{\alpha_N} = \mathbb{C}\cdot 1_N$. (We could also have avoided the use of this strong invariance formula, using instead a 'Heisenberg algebra' type of argument as in Proposition 7.2.17.)

Finally, consider the Galois isometry

$$\tilde{G} : \mathscr{L}^2(N) \otimes \mathscr{L}^2(N) \to \mathscr{L}^2(M) \otimes \mathscr{L}^2(N).$$

Then it is easily seen to coincide with the unitary map $\Sigma(W^2_{12})^*\Sigma$. Hence (N, α_N) is a right Galois object for M.

□

This means that if M and P are von Neumann algebraic quantum groups, and $\hat{Q}$ is a linking von Neumann algebraic quantum groupoid between $\widehat{M}$ and $\hat{P}$, we have a canonical way to construct a right Galois object N for M from it, and we will write $N = \mathrm{Gal}_r(\hat{Q})$. Conversely, by the results of the previous section, if we have a right Galois object for M, then we can construct from it in a canonical way a linking von Neumann algebraic quantum groupoid, which we will write for the moment with an extra ˜, i.e. as $\tilde{\hat{Q}}$, and we will also write $\tilde{\hat{Q}} = \mathrm{LQG}(N)$. Following carefully the iterate of these constructions, one can conclude that in fact $\tilde{\hat{Q}} = \mathrm{LQG}(\mathrm{Gal}_r(Q))$ is a linking von Neumann algebraic quantum groupoid between $\widehat{M} = \hat{\pi}^2_{22}(\widehat{M})$ and $\hat{\pi}^2_{11}(\hat{P})$, using identity maps. Then by identifying $\hat{P}$ with $\hat{\pi}^2_{11}(\hat{P})$ via $\hat{\pi}^2_{11}$, we get that $\tilde{\hat{Q}}$ is a linking von Neumann algebraic quantum groupoid between $\widehat{M}$ and $\hat{P}$. Then $\pi^{\hat{Q},2}$ is an isomorphism between the linking von Neumann algebraic quantum groupoids $\hat{Q}$ and $\tilde{\hat{Q}}$ between $\widehat{M}$ and $\hat{P}$.

Conversely, it is easy to see that if N is a right M-Galois object, then $N = \mathrm{Gal}_r(\mathrm{LQG}(N))$ as right M-Galois objects. Hence:

Corollary 7.4.7. *There is a natural one-to-one correspondence between right Galois objects and linking von Neumann algebraic quantum groupoids.*

We have to warn however, that this correspondence *does not* pass to isomorphism classes (for the issue of isomorphism questions, see for example the remarks made in section 1.1.2). We will return to this issue in the next subsection.

7.4.3 Bi-Galois objects

Definition 7.4.8. *Let M and P be von Neumann algebraic quantum groups. A P-M-*bi-Galois object *(or* bi-Galois object between M and P*) consists of a triple (N, γ_N, α_N) such that (N, α_N) (resp. (N, γ_N)) is a right (resp. left) Galois object for M (resp. P), and such that α_N and γ_N commute. We call M and P* monoidally W*-co-Morita equivalent *if there exists a P-M-bi-Galois object.*

Proposition 7.4.9. *Let Q be a co-linking von Neumann algebraic quantum groupoid. Then (N, γ_N, α_N) is a P-M-bi-Galois object and (O, γ_O, α_O) an M-P-bi-Galois object.*

Proof. The four coactions which appear all induce left or right Galois object structures, by Proposition 7.4.6 and symmetry arguments. So we only have to see if γ_N and α_N commute. But this is immediate from the (piecewise) coassociativity of Δ_Q. □

We prove a proposition concerning the reconstruction of a bi-Galois object from its associated right Galois object.

Proposition 7.4.10. *Let $(N, \tilde{\gamma}_N, \alpha_N)$ be a $\tilde{P}$-M-bi-Galois object for von Neumann algebraic quantum groups M and $\tilde{P}$. Denote by P the reflection of M along N, and by (N, γ_N, α_N) the associated bi-Galois object. Then the canonical normal left representation of $\hat{\tilde{P}}$ on $\mathscr{L}^2(N)$ provides an isomorphism $\hat{\Phi} : \hat{\tilde{P}} \rightarrow \hat{P}$ of von Neumann algebraic quantum groups, such that, denoting by Φ the dual isomorphism between $\tilde{P}$ and P,*

$$\tilde{\gamma}_N = (\Phi \otimes \iota_N)\gamma_N.$$

Proof. Let $\varphi_N = (\iota_N \otimes \varphi_M)\alpha_N$. Choose a state $\omega \in N_*$. Then for $x \in \mathscr{M}^+_{\varphi_N}$ and a non-zero $\omega' \in \tilde{P}^+_*$, we have that

$$\begin{aligned}
\varphi_N((\omega' \otimes \iota_N)\tilde{\gamma}_N(x)) &= \varphi_M((\omega' \otimes \omega \otimes \iota_M)((\iota \otimes \alpha_N)\tilde{\gamma}_N(x))) \\
&= \varphi_M((\omega' \otimes \omega \otimes \iota_M)((\tilde{\gamma}_N \otimes \iota_M)\alpha_N(x))) \\
&= \varphi_N(x)(\omega' \otimes \omega)(\tilde{\gamma}_N(1_N)) \\
&= \varphi_N(x)\omega'(1_{\tilde{P}}).
\end{aligned}$$

By the uniqueness of an invariant nsf weight for a left Galois object, we conclude that φ_N coincides with the invariant nsf weight as constructed from $(N, \tilde{\gamma}_N)$ (up to a scalar). Similarly for ψ_N, which we define as $(\psi_{\tilde{P}} \otimes \iota_N)\tilde{\gamma}_N$, and for which we then show that it is α_N-invariant.

Now let $\tilde{H}_{\tilde{P}}$ be the Galois unitary for $(N, \tilde{\gamma}_N)$, and $\tilde{H}_P$ the one for (P, γ_N). Then $\tilde{H}_P \tilde{H}_{\tilde{P}}^*$ is a unitary in $B(\mathscr{L}^2(\tilde{P}), \mathscr{L}^2(P)) \otimes N$ by Lemma 6.4.10. We prove now that $\tilde{H}_{\tilde{P}}$ satisfies a pentagonal identity with respect to U, the unitary implementation of α_N, namely

$$(\tilde{H}_{\tilde{P}})_{12} U_{13} U_{23} = U_{23} (\tilde{H}_{\tilde{P}})_{12}.$$

Indeed, this is an easy verification, using the fact that $\tilde{\gamma}_N$ commutes with α_N: for $x, y \in \mathscr{N}_{\psi_N}$ and m in (for example) $\mathscr{N}_{\psi_M}$, we have

$$\begin{aligned}
&((\tilde{H}_{\tilde{P}})_{12} U_{13} U_{23})(\Gamma_N(x) \otimes \Gamma_N(y) \otimes \Gamma_M(m)) \\
&= (\Gamma_N \otimes \Gamma_N \otimes \Gamma_M)((\tilde{\gamma}_N \otimes \iota_M)(\alpha_N(x)) \cdot (1_{\tilde{P}} \otimes \alpha_N(y))(1_{\tilde{P}} \otimes 1_N \otimes m)) \\
&= (\Gamma_N \otimes \Gamma_N \otimes \Gamma_M)((\iota_{\tilde{P}} \otimes \alpha_N)(\tilde{\gamma}_N(x)) \cdot (1_{\tilde{P}} \otimes \alpha_N(y))(1_{\tilde{P}} \otimes 1_N \otimes m)) \\
&= U_{23} (\tilde{H}_{\tilde{P}})_{12} (\Gamma_N(x) \otimes \Gamma_N(y) \otimes \Gamma_M(m)),
\end{aligned}$$

which is a rather careless calculation, easy to make more rigorous.

Since the same pentagonal identity holds for $\tilde{H}_P$, we have that

$$(\tilde{H}_P \tilde{H}_{\tilde{P}}^*)_{12} U_{23} = U_{23} (\tilde{H}_P \tilde{H}_{\tilde{P}}^*)_{12},$$

which implies that $\tilde{H}_P \tilde{H}_{\tilde{P}}^* \in B(\mathscr{L}^2(\tilde{P}), \mathscr{L}^2(P)) \otimes (N \cap \widehat{P})$. But $(N \cap \widehat{P})' = (N' \cup \widehat{P}')''$, which equals the whole of $B(\mathscr{L}^2(N))$. Hence $\tilde{H}_P \tilde{H}_{\tilde{P}}^* = v \otimes 1$ for some unitary $v : \mathscr{L}^2(P) \to \mathscr{L}^2(\tilde{P})$.

Now since $\tilde{H}_P = (v \otimes 1) \tilde{H}_{\tilde{P}}$, the second item of Lemma 7.2.6 implies that, denoting $\widehat{\tilde{O}}'$ the space of left $\widehat{\tilde{P}}$-intertwiners $\mathscr{L}^2(N) \to \mathscr{L}^2(\tilde{P})$, and similarly by $\widehat{O}'$ the space of left $\widehat{P}$-intertwiners $\mathscr{L}^2(N) \to \mathscr{L}^2(P)$,

$$\widehat{\tilde{O}}' \to \widehat{O}' : x \to vx$$

is an isomorphism. Then clearly

$$\widehat{\Phi}' : \widehat{\tilde{P}}' \to \widehat{P}' : x \to vxv^*$$

is also an isomorphism.

We show now that $\widehat{\Phi}'$ preserves the comultiplication. By the pentagonal identity for $\tilde{H}_{\tilde{P}}$, we have

$$(V_{\tilde{P}})_{12}(\tilde{H}_{\tilde{P}})_{13}(\tilde{H}_{\tilde{P}})_{23} = (\tilde{H}_{\tilde{P}})_{23}(\tilde{H}_{\tilde{P}})_{12}.$$

Using the similar identity for $\tilde{H}_P$ and the fact that $\tilde{H}_P = (v \otimes 1)\tilde{H}_{\tilde{P}}$, we conclude that

$$(v \otimes v)V_{\tilde{P}}(v^* \otimes v^*) = V_P,$$

which immediately implies that $\widehat{\Phi}'$ preserves the comultiplication.

Now we show that the inverse of the dual $\Phi : P \to \tilde{P}$ of $\widehat{\Phi}'$, uniquely determined by the identity $(\widehat{\Phi}' \otimes \Phi^{-1})(V_{\tilde{P}}) = V_P$, intertwines $\tilde{\gamma}_N$ and γ_N in the manner indicated in the proposition. But clearly, $\Phi = \mathrm{Ad}(v^*)$. So

$$\begin{aligned}
(\iota_{\widehat{\tilde{O}'}} \otimes ((\Phi^{-1} \otimes \iota_N)\tilde{\gamma}_N))(\tilde{H}_{\tilde{P}}) &= (\iota_{\widehat{\tilde{O}'}} \otimes \Phi^{-1} \otimes \iota_N)((V_{\tilde{P}})_{12}(\tilde{H}_{\tilde{P}})_{13}) \\
&= (v^* \otimes 1 \otimes 1)(V_P)_{12}(v \otimes 1 \otimes 1)(\tilde{H}_{\tilde{P}})_{13} \\
&= (v^* \otimes 1 \otimes 1)(V_P)_{12}(\tilde{H}_P)_{13} \\
&= (v^* \otimes 1 \otimes 1)(\iota_{\widehat{O}'} \otimes \gamma_N)(\tilde{H}_P) \\
&= (\iota_{\widehat{\tilde{O}'}} \otimes \gamma_N)(\tilde{H}_{\tilde{P}}).
\end{aligned}$$

Since the second leg of $\tilde{H}_{\tilde{P}}$ is σ-weakly dense inside N, the intertwining property follows.

□

Hence any bi-Galois object can be recuperated from its associated right Galois object, and in particular, two von Neumann algebraic quantum groups are monoidally W*-co-Morita equivalent iff their duals are comonoidally W*-Morita equivalent. Observe however that again, the isomorphism class of a bi-Galois object is not determined by the isomorphism class of the associated right Galois object. In fact, we have the following proposition, which is a straightforward analogue of a result of [71].

Proposition 7.4.11. *Let M and P be von Neumann algebraic quantum groups, let (N, α_N) be a right M-Galois object, and let (N, γ_N, α_N) and $(N, \tilde{\gamma}_N, \alpha_N)$ be two P-M-bi-Galois objects. Then there exists an automorphism Φ_P of the von Neumann algebraic quantum group P such that $(\Phi_P \otimes \iota)\gamma_N = \tilde{\gamma}_N$. Moreover, the two bi-Galois objects will be isomorphic iff there exists a group-like unitary $u \in \widehat{P}$ such that $\widehat{\Phi}_P = Ad(u)$.*

Proof. The first statement follows immediately from the previous proposition.

Now suppose that (N, γ_N, α_N) is a P-M-bi-Galois object, where we can suppose that P is the reflection of M across N. Suppose that Φ_P is an automorphism of the von Neumann algebraic quantum group P, such that (N, γ_N, α_N) and $(N, (\Phi_P \otimes \iota_N)\gamma_N, \alpha_N)$ are isomorphic, that is, that there exists an isomorphism $\Phi_N : N \to N$ of von Neumann algebras, such that

$$(\Phi_N \otimes \iota) \circ \alpha_N = \alpha_N \circ \Phi_N,$$

$$(\Phi_P \otimes \Phi_N) \circ \gamma_N = \gamma_N \circ \Phi_N.$$

Then it follows by the first commutation that $\varphi_N \circ \Phi_N = \varphi_N$. Define a unitary

$$u : \mathscr{L}^2(N) \to \mathscr{L}^2(N) : \Lambda_{\varphi_N}(x) \to \Lambda_{\varphi_N}(\Phi_N(x)), \qquad x \in \mathscr{N}_{\varphi_N}.$$

Then an easy calculation, using again the first commutation, shows that

$$\widehat{\pi}'_{\alpha_N}((\iota \otimes \omega)(V_M))u = u\widehat{\pi}'_{\alpha_N}((\iota \otimes \omega)(V_M)), \qquad \omega \in M_*.$$

Hence $u \in \widehat{\theta}_{\alpha_N}(\widehat{M})'$. But $\widehat{\theta}_{\alpha_N}(\widehat{M})' = \widehat{P}$. Moreover, if $\tilde{G}$ is the Galois unitary for α_N, then an easy calculation shows that

$$\tilde{G}(u \otimes u) = (1 \otimes u)\tilde{G},$$

so that, since $\Delta_{\widehat{P}}(x) = \tilde{G}^*(1 \otimes x)\tilde{G}$ by definition,

$$\Delta_{\widehat{P}}(u) = u \otimes u,$$

i.e. u is a group-like element of $(\widehat{P}, \Delta_{\widehat{P}})$.

We now show that u^* implements $\widehat{\Phi}_P$, the dual of Φ_P. This is again easy: if U_P is the unitary implementation of γ_N, and $\omega \in P_*$, $x \in \mathscr{N}_{\varphi_N}$, then

$$\begin{aligned}(\omega \otimes \iota)(U_P)u\Lambda_N(x) &= \Lambda_N((\omega \otimes \iota)\gamma_N(\Phi_N(x))) \\ &= \Lambda_N(\Phi_N((\omega \circ \Phi_P \otimes \iota)\gamma_N(x))) \\ &= u(\omega \circ \Phi_P \otimes \iota)(U_P)\Lambda_N(x),\end{aligned}$$

which implies $(\Phi_P \otimes \mathrm{Ad}(u))(W_P^*) = W_P^*$, and so $\widehat{\Phi}_P = \mathrm{Ad}(u^*)$. So Φ_P is necessarily a co-inner automorphism (in the sense that its dual is inner by a group-like element).

Conversely, it is not difficult to see that if Φ_P is co-inner by a group-like element $u^* \in \widehat{P}$ (so $\widehat{\Phi}_P(x) = u^*xu$ for $x \in \widehat{P}$), then

$$N \to N : x \to uxu^*$$

will be a well-defined isomorphism from (N, γ_N, α_N) to $(N, (\Phi_P \otimes \iota)\gamma_N, \alpha_N)$. First of all, if $x \in N$, then uxu^* will end up in N by the biduality theorem: if $\widehat{\gamma_N}$ is the dual right coaction of γ_N on $B(\mathscr{L}^2(N)) \cong P \underset{\gamma_N}{\ltimes} N$, then for $x \in N$,

$$\begin{aligned}
\widehat{\gamma_N}(uxu^*) &= \Delta_{\widehat{P}}(u)(x \otimes 1)\Delta_{\widehat{P}}(u^*) \\
&= (u \otimes u)(x \otimes 1)(u^* \otimes u^*) \\
&= (uxu^* \otimes 1).
\end{aligned}$$

But then $uxu^* \in (P \underset{\gamma_N}{\ltimes} N)^{\gamma_N}$, which is exactly N.

Since u commutes elementwise with $\widehat{\theta}_{\alpha_N}(\widehat{M})$, we will have

$$(u \otimes 1)U = U(u \otimes 1),$$

where U is the unitary implementation of α_N, hence

$$(\Phi_N \otimes \iota) \circ \alpha_N = \alpha_N \circ \Phi_N.$$

And since $u^*xu = \widehat{\Phi}_P(x)$ for $x \in \widehat{P}$, we will have $(1 \otimes u^*)U_P = (\Phi_P \otimes \iota)(U_P)(1 \otimes u^*)$, where U_P is the unitary implementation of γ_N, and hence

$$(\Phi_P \otimes \Phi_N) \circ \gamma_N = \gamma_N \circ \Phi_N.$$

□

From the previous proof, it is also easily seen that, in the notation of the previous proposition, the set of isomorphisms from (N, γ_N, α_N) to $(N, (\Phi_P \otimes \iota_N)\gamma_N, \alpha_N)$ is parametrized by the set

$$\{u \in \widehat{P} \mid u \text{ grouplike and implementing } \widehat{\Phi}_P\}.$$

Indeed: one further only has to observe that $\widehat{P} \cap N' = \mathbb{C}$.

7.4.4 Further structure of (co-)linking quantum groupoids

In reconstructing a linking von Neumann algebraic quantum groupoid $(\widehat{Q}, e)$ from a right M-Galois object N, we introduced some auxiliary structures, such as δ_N, P_N,... On the other hand, a von Neumann algebraic (co-)linking quantum groupoid, which is a measured quantum groupoid, comes with some structure of its own, such as a scaling group, a modular element, a scaling operator, ... We show here that both these structures are the same.

So let $(\widehat{Q}, e)$ be a linking von Neumann algebraic quantum groupoid and Q the associated co-linking von Neumann algebraic quantum groupoid. Let N be the associated right M-Galois object, with Galois unitary $\tilde{G}$, using notation as before.

First, we give some more information about the unitaries $\widehat{W}_{ik}^j$ pertaining to the linking quantum groupoid. By construction, we have that

$$\widehat{W}_{22}^2 = W_{\widehat{M}}, \qquad \widehat{W}_{12}^2 = \tilde{G}.$$

By Proposition 7.3.6, we also have that

$$\widehat{W}_{21}^2 = (J_N \otimes J_{\widehat{N}})\tilde{G}^*(J_M \otimes J_{\widehat{O}}).$$

We have the further identity

$$\widehat{W}_{22}^1 = (J_{\widehat{N}} \otimes J_{\widehat{M}})U(J_{\widehat{O}} \otimes J_{\widehat{M}}),$$

where U is the unitary implementation of α_N. For this, use for example that

$$\widehat{\pi}_{22}^1(m) = J_{\widehat{N}}\widehat{\pi}'_{\alpha_N}(J_{\widehat{M}}m^*J_{\widehat{M}})^*J_{\widehat{O}}$$

when $m \in \widehat{M}$, and that

$$\begin{aligned}
&(J_{\widehat{N}} \otimes J_{\widehat{M}})U(J_{\widehat{O}} \otimes J_{\widehat{M}}) \\
&\quad = (J_{\widehat{N}} \otimes J_{\widehat{M}})(\widehat{\pi}'_{\alpha_N} \otimes \iota)(V_M)(J_{\widehat{O}} \otimes J_{\widehat{M}}) \\
&\quad = (J_{\widehat{N}} \otimes J_{\widehat{M}})(\widehat{\pi}'_{\alpha_N} \otimes \iota)((J_{\widehat{M}} \otimes J_{\widehat{M}})W_{\widehat{M}}(J_{\widehat{M}} \otimes J_{\widehat{M}}))(J_{\widehat{O}} \otimes J_{\widehat{M}}) \\
&\quad = (\widehat{\pi}_{22}^1 \otimes \iota)(W_{\widehat{M}}).
\end{aligned}$$

Then the stated equality follows from Lemma 7.4.5. This gives us descriptions for four of the maps $\widehat{W}_{ij}^k$ (and *three* of the maps $\widehat{W}_{ij}$) constituting $W_{\widehat{Q}}$ in terms of the associated right Galois object (N, α_N). The other four can then be described in terms of the Galois map, unitary corepresentation and multiplicative unitary of the associated left Galois object (N, γ_N). *Copies* of these four maps however can also be obtained directly by using only the right Galois object. We will make this clear later on for the multiplicative unitary of $\widehat{P}$ (see Proposition 7.4.18).

The *modular operator* $\nabla_{\varphi_{\hat{Q}}}$ *for* $\varphi_{\hat{Q}}$ is easy to describe, since it is just the modular operator for a balanced weight, whose structure we have already described. So

$$\nabla^{it}_{\varphi_{\hat{Q}}} = \nabla^{it}_{\varphi_{\hat{P}}} \oplus \nabla^{it}_{\hat{O}} \oplus \nabla^{it}_{\hat{N}} \oplus \nabla^{it}_{\varphi_{\widehat{M}}},$$

and $\nabla_{\varphi_{\widehat{M}}}$ is the modular operator for $\varphi_{\widehat{M}}$, while $\nabla_{\hat{N}}$ is the spatial derivative of $\varphi_{\hat{P}}$ with respect to $\varphi'_{\widehat{M}}$. The fact that $\nabla_{\hat{N}}$ coincides with the map $\nabla_{\hat{N}}$ constructed from the right Galois object is obvious, by construction. Hence the modular one-parametergroup $\sigma_t^{\varphi_{\hat{Q}}}$ can be written in the well-known form

$$\sigma_t^{\varphi_{\hat{Q}}}(\begin{pmatrix} x & y \\ w & z \end{pmatrix}) = \begin{pmatrix} \sigma_t^{\varphi_{\hat{P}}}(x) & \sigma_t^{\varphi_{\hat{P}},\varphi_{\widehat{M}}}(y) \\ \sigma_t^{\varphi_{\widehat{M}},\varphi_{\hat{P}}}(w) & \sigma_t^{\varphi_{\widehat{M}}}(z) \end{pmatrix}.$$

The *modular operator* ∇_{φ_Q} *for* φ_Q splits up into a direct sum:

$$\nabla^{it}_{\varphi_Q} = \nabla^{it}_{\varphi_P} \oplus \nabla^{it}_{\varphi_O} \oplus \nabla^{it}_{\varphi_N} \oplus \nabla^{it}_{\varphi_M}.$$

This is obvious, as the weight φ_Q is a direct sum of weights, and then ∇_{φ_N} is just the modular operator for the δ_M-invariant nsf weight φ_N of the associated right Galois object. The corresponding form of the modular one-parametergroup is then easily derived.

The *modular element* $\delta_{\hat{Q}}$ *of* $(\widehat{Q}, e)$ will be of the form

$$\delta_{\hat{Q}} = \begin{pmatrix} \delta_{\hat{P}} & 0 \\ 0 & \delta_{\widehat{M}} \end{pmatrix},$$

with $\delta_{\hat{P}}$ and $\delta_{\widehat{M}}$ the modular elements for resp. $\widehat{P}$ and $\widehat{M}$. Again, this is easy, since $\delta_{\hat{Q}}$ is uniquely characterized by the identity $\psi_{\hat{Q}} = \varphi_{\hat{Q}}(\delta^{1/2}_{\hat{Q}} \cdot \delta^{1/2}_{\hat{Q}})$, where $\psi_{\hat{Q}}$ and $\varphi_{\hat{Q}}$ are just the balanced weights of $\psi_{\hat{P}}$ and $\psi_{\widehat{M}}$, resp. $\varphi_{\hat{P}}$ and $\varphi_{\widehat{M}}$.

The *modular element* δ_Q *of* Q can be written as

$$\delta_Q = \delta_P \oplus \delta_O \oplus \delta_N \oplus \delta_M,$$

with δ_P the modular element of P and δ_M the one for M (with respect to the fixed left and right invariant nsf weights on M and P by restricting the ones on Q). Then δ_N will coincide with the modular element for N introduced in Definition 7.2.10, possibly up to a positive scalar. This follows from the fact that there is a unique α_N-invariant nsf weight on N, *up to a positive scalar*, and the fact that, *once a right α_N-invariant nsf weight on N is fixed* (such as ψ_N), then δ_N is uniquely determined by the property that $\nu_M^{it^2/2}\delta_N^{it}$ is the cocycle derivative of ψ_N w.r.t. φ_N. Note that, in the construction of a linking von Neumann algebraic quantum groupoid from a right Galois object, different scalings of δ_N will correspond to different scalings of the left invariant weight $\varphi_{\widehat{P}}$ on $\widehat{P}$. In particular, *there seems to be no canonical choice of invariant weight on* $\widehat{P}$ in terms of the right Galois object (N, α_N).

By using that *the scaling group $\tau_t^{\widehat{Q}}$ of $(\widehat{Q}, e)$* is implemented by $\nabla_{\varphi_Q}^{it}$ (see theorem 3.10.(vii) of [30]), we find that it is of the form

$$\tau_t^{\widehat{Q}}(\begin{pmatrix} x & y \\ w & z \end{pmatrix}) = \begin{pmatrix} \tau_t^{\widehat{P}}(x) & \tau_t^{\widehat{N}}(y) \\ \tau_t^{\widehat{O}}(w) & \tau_t^{\widehat{M}}(z) \end{pmatrix},$$

where $\tau_t^{\widehat{P}}$ and $\tau_t^{\widehat{M}}$ are the scaling groups of resp. $\widehat{P}$ and $\widehat{M}$, and where $\tau_t^{\widehat{N}}$ and $\tau_t^{\widehat{O}}$ are certain one-parameter transformation groups of resp. $\widehat{N}$ and $\widehat{O}$.

On the other hand, by using that *the scaling group τ_t^Q of Q* is implemented by $\nabla_{\varphi_{\widehat{Q}}}^{it}$, we find that

$$\tau_t^Q = \tau_t^P \oplus \tau_t^O \oplus \tau_t^N \oplus \tau_t^M,$$

where τ_t^M and τ_t^P are the scaling groups of respectively M and P, and where τ_t^N is the scaling group on N, as introduced in Definition 7.2.2. Since $\nabla_{\widehat{O}}^{it} = J_{\widehat{N}}\nabla_{\widehat{N}}^{it}J_{\widehat{O}}$, we also find that $\tau_t^O(x) = J_{\widehat{N}}\tau_t^N(J_{\widehat{O}}xJ_{\widehat{N}})J_{\widehat{O}}$ for $x \in O$.

The *unitary antipode $R_{\widehat{Q}}$ of $(\widehat{Q}, e)$* will be implemented by J_Q, which is just the direct sum $J_P \oplus J_O \oplus J_N \oplus J_M$. Therefore,

$$R_{\widehat{Q}}(\begin{pmatrix} x & y \\ w & z \end{pmatrix}) = \begin{pmatrix} R_{\widehat{P}}(x) & R_{\widehat{O}}(w) \\ R_{\widehat{N}}(y) & R_{\widehat{M}}(z) \end{pmatrix},$$

where $R_{\widehat{P}}$ and $R_{\widehat{M}}$ are the unitary antipodes of resp. $\widehat{P}$ and $\widehat{M}$, and where $R_{\widehat{N}}(y)$ for example equals $J_M y^* J_N$.

Since the *unitary antipode* R_Q *of* Q is implemented by

$$J_{\widehat{Q}} : \bigoplus_{ij} \mathscr{L}^2(\widehat{Q}_{ij}) \to \bigoplus_{ij} \mathscr{L}^2(\widehat{Q}_{ij}) :$$

$$\begin{pmatrix} \xi_{11} & \xi_{21} & \xi_{12} & \xi_{22} \end{pmatrix} \to \begin{pmatrix} J_{\widehat{P}}\xi_{11} & J_{\widehat{N}}\xi_{12} & J_{\widehat{O}}\xi_{21} & J_{\widehat{M}}\xi_{22} \end{pmatrix},$$

we also have that

$$R_Q(x \oplus z \oplus y \oplus w) = R_P(x) \oplus R_N(y) \oplus R_O(z) \oplus R_M(w),$$

where $x \in P, z \in O, y \in N, w \in M$, where R_P and R_M are the unitary antipodes of respectively P and M, and where for example $R_N(y) = J_{\widehat{N}} y^* J_{\widehat{O}}$.

Finally, note that by Lemma 7.2.11, we have that $\sigma_t^{\varphi_N}(\delta_N^{is}) = \nu_M^{ist}\delta_N^{is}$, while, since ψ_N plays the role of φ_N for the right P^{cop}-Galois object $(N, \gamma_N^{\mathrm{op}})$, we also have $\sigma_t^{\psi_N}(\delta_N^{-is}) = \nu_{P^{\mathrm{cop}}}^{ist}\delta_N^{-is}$. Since $\sigma_t^{\psi_N}(\delta_N^{is}) = \sigma_t^{\varphi_N}(\delta_N^{is})$, we conclude that $\nu_{P^{\mathrm{cop}}} = \nu_M^{-1}$, and then $\nu_P = \nu_M$. Hence:

Corollary 7.4.12. *If M and P are monoidally W^*-co-Morita equivalent von Neumann algebraic quantum groups, then they have the same scaling constant.*

It is then clear that the *scaling operators* of the measured quantum groupoids Q and $\widehat{Q}$ (see Theorem 3.8.vi) of [30]) are scalar multiples of the unit.

7.4.5 Multiplicative unitaries

We again fix a linking von Neumann algebraic quantum groupoid $\widehat{Q}$.

Let $W_Q = \Sigma W_{\widehat{Q}}^* \Sigma$ be the left regular multiplicative partial isometry associated with its dual co-linking von Neumann algebraic quantum groupoid. We have the following formulas, which are immediate consequences of the pentagonal identity for W_Q:

Lemma 7.4.13. *1. For all $i, j, k, l \in \{1, 2\}$, we have*

$$(W_{ij}^k)_{12}(W_{ij}^l)_{13}(W_{jk}^l)_{23} = (W_{ik}^l)_{23}(W_{ij}^k)_{12}$$

as operators

$$\mathscr{L}^2(Q_{ij}) \otimes \mathscr{L}^2(Q_{jk}) \otimes \mathscr{L}^2(Q_{kl}) \to \mathscr{L}^2(Q_{ij}) \otimes \mathscr{L}^2(Q_{ik}) \otimes \mathscr{L}^2(Q_{il}).$$

2. For all $i, j, k \in \{1, 2\}$ and $x \in Q_{ij}$, we have $\Delta_{ij}^k(x) = (W_{ik}^j)^*(1 \otimes x)W_{ik}^j$.

3. *For all $i,j,k,l \in \{1,2\}$, we have* $(\Delta_{ik}^{j} \otimes \iota)(W_{ik}^{l}) = (W_{ij}^{l})_{13}(W_{jk}^{l})_{23}$.

Of course, we can also consider a right multiplicative partial isometry V_Q. This will split up into unitaries

$$V_{kj}^{i} : \mathscr{L}^2(Q_{ij}) \otimes \mathscr{L}^2(Q_{kj}) \to \mathscr{L}^2(Q_{ik}) \otimes \mathscr{L}^2(Q_{kj}),$$

and then

Lemma 7.4.14. 1. *For all $i,j,k,l \in \{1,2\}$, we have*

$$(V_{jk}^{i})_{12}(V_{kl}^{i})_{13}(V_{kl}^{j})_{23} = (V_{kl}^{j})_{23}(V_{jl}^{i})_{12}$$

as operators

$$\mathscr{L}^2(Q_{il}) \otimes \mathscr{L}^2(Q_{jl}) \otimes \mathscr{L}^2(Q_{kl}) \to \mathscr{L}^2(Q_{ij}) \otimes \mathscr{L}^2(Q_{jk}) \otimes \mathscr{L}^2(Q_{kl}).$$

2. *For all $i,j,k \in \{1,2\}$ and $x \in Q_{ij}$, we have* $\Delta_{ij}^{k}(x) = V_{kj}^{i}(x \otimes 1)(V_{kj}^{i})^*$.

3. *For all $i,j,k,l \in \{1,2\}$, we have* $(\iota \otimes \Delta_{jl}^{k})(V_{jl}^{i}) = (V_{jk}^{i})_{12}(V_{kl}^{i})_{13}$.

We now introduce the notion of a quantum torsor (which really only depends upon the isomorphism class of the von Neumann algebraic (co-)linking quantum groupoid, but which can then of course also be associated naturally to any right Galois object).

Definition 7.4.15. *If Q is a co-linking von Neumann algebraic quantum groupoid, then the associated* quantum torsor *is the couple (N,Θ), where Θ is the map*

$$\Theta : N \to N \otimes O \otimes N : x \to (\iota_N \otimes \beta_M)\alpha_N(x) = (\beta_P \otimes \iota_N)\gamma_N(x).$$

Note that in the previous definition, we should identify O with $\overline{N}$ (or N^{op}, or N') by sending x to $\overline{R_Q(x)^*}$ (or $(R_Q(x))^{\mathrm{op}}$, or $C_N(R_Q(x))$), as to make the notion of quantum torsor involve only one von Neumann algebra. But since we will not define a (von Neumann algebraic) quantum torsor independently, we will just keep using the O-notation.

In the following lemma, we construct a multiplicative unitary for this quantum torsor.

Lemma 7.4.16. *For all $x, z \in \mathscr{N}_{\varphi_N}$ and $y \in \mathscr{N}_{\varphi_O}$, we have $\Theta(z)(x \otimes y \otimes 1) \in \mathscr{N}_{\varphi_N \otimes \varphi_O \otimes \varphi_N}$, and*

$$(\Lambda_N \otimes \Lambda_O \otimes \Lambda_N)(\Theta(z)(x \otimes y \otimes 1)) = (W_{21}^2)_{23}^*(W_{12}^2)_{13}^*(\Lambda_N(x) \otimes \Lambda_O(y) \otimes \Lambda_N(z)).$$

Proof. This follows immediately by the definition of W_Q:

$$\begin{aligned} &(W_{21}^2)_{23}^*(W_{12}^2)_{13}^*(\Lambda_{\varphi_{12}}(x) \otimes \Lambda_{\varphi_{21}}(y) \otimes \Lambda_{\varphi_{12}}(z)) \\ &= (W_{21}^2)^*(\Lambda_{\varphi_{12}} \otimes \Lambda_{\varphi_{21}} \otimes \Lambda_{\varphi_{22}})((\Delta_{12}^2(z))_{13}(x \otimes y \otimes 1)) \\ &= (\Lambda_{\varphi_{12}} \otimes \Lambda_{\varphi_{21}} \otimes \Lambda_{\varphi_{12}})(((\iota \otimes \Delta_{22}^1)\Delta_{12}^2)(z)(x \otimes y \otimes 1)) \\ &= (\Lambda_N \otimes \Lambda_O \otimes \Lambda_N)(\Theta(z)(x \otimes y \otimes 1)). \end{aligned}$$

□

We define $W^\Theta := (W_{12}^2)_{13}(W_{21}^2)_{23}$. It satisfies a pentagon identity:

Proposition 7.4.17. *Let (N, Θ) be a quantum torsor. Then the following commutation relation holds:*

$$(W^\Theta)_{123}(W^\Theta)_{125}(W^\Theta)_{345} = (W^\Theta)_{345}(W^\Theta)_{123}.$$

Proof. Taking the adjoints of these expressions, the equality easily follows by the formula of the previous lemma. □

We can use W^Θ to provide a different multiplicative unitary for P. Denote $\mathscr{H} = \mathscr{L}^2(N) \otimes \mathscr{L}^2(O)$.

Proposition 7.4.18. *We have that $W^\Theta = (\beta_P \otimes \hat{\pi}_{11}^2)(W_P)$, and $\tilde{W}_P := (W^\Theta \otimes 1)$, seen as an operator on $\mathscr{H} \otimes \mathscr{H}$, is a multiplicative unitary for the von Neumann algebraic quantum group P.*

Proof. It follows from Lemma 7.4.5 that $(\iota \otimes \hat{\pi}_{11}^2)(W_P) = W_{11}^2$. From Lemma 7.4.13 it follows that

$$\begin{aligned} (\beta_P \otimes \iota)(W_{11}^2) &= (W_{12}^2)_{13}(W_{21}^2)_{23} \\ &= W^\Theta. \end{aligned}$$

By the pentagon equation for W^Θ, it follows easily that $\tilde{W}_P$ is a multiplicative unitary. Then also

$$\begin{aligned} &(\tilde{W}_{P,1234}^*((\beta_P \otimes \hat{\pi}_{11}^2)(W_P))_{345}\tilde{W}_{P,1234}) \otimes 1 \\ &= \tilde{W}_{P,1234}^*\tilde{W}_{P,3456}\tilde{W}_{P,1234} \\ &= \tilde{W}_{P,1256}\tilde{W}_{P,3456} \\ &= (((\beta_P \otimes \hat{\pi}_{11}^2)(W_P))_{125}((\beta_P \otimes \hat{\pi}_{11}^2)(W_P))_{345}) \otimes 1 \\ &= (\beta_P \otimes \beta_P \otimes \hat{\pi}_{11}^2)((W_P)_{13}(W_P)_{23}) \otimes 1 \\ &= (((\beta_P \otimes \beta_P) \circ \Delta_P) \otimes \hat{\pi}_{11}^2)(W_P) \otimes 1, \end{aligned}$$

from which it follows that $\tilde{W}_P$ is a multiplicative unitary for $(\beta_P(P), (\beta_P \otimes \beta_P) \circ \Delta_P \circ \beta_P^{-1}) \cong (P, \Delta_P)$.

□

Since $W_{12}^2 = \Sigma\tilde{G}^*\Sigma$, where $\tilde{G}$ is the Galois unitary for the associated right Galois object N, and, since we have already argued that $W_{21}^2 = \Sigma\tilde{G}_J\Sigma$, where $\tilde{G}_J$ denotes the operator $(J_M \otimes J_{\hat{N}})\tilde{G}(J_N \otimes J_{\hat{O}})$, this means that the multiplicative unitary of the von Neumann algebraic quantum group P can be constructed directly from the Galois unitary $\tilde{G}$ and the modular conjugations J_N and J_M associated with the right Galois object N (since the restrictions $J_{\hat{N}}$ and $J_{\hat{O}}$ of $J_{\hat{Q}}$ are just formal constructions, see the remark after the following proposition). In fact, we can use this to reconstruct the von Neumann algebraic quantum group P from N in a direct manner (without passing to the dual $\hat{P}$), which is more in line with the method in Hopf algebra theory (but of course, for us this is rather an *a posteriori* construction!).

Proposition 7.4.19. *The von Neumann algebra*

$$\tilde{P} = \{z \in N \otimes O \mid (\iota \otimes \gamma_O)(z) = (\alpha_N \otimes \iota)(z)\},$$

together with the comultiplication

$$\Delta_{\tilde{P}}(x) = (\Theta \otimes \iota)(x), \qquad x \in \tilde{P},$$

will be a well-defined Hopf-von Neumann algebra, isomorphic to the von Neumann algebraic quantum group P by the map β_P.

Proof. It is not difficult to see that $\tilde{P}$ is a well-defined coinvolutive Hopf-von Neumann algebra, using the various coassociativity relations between the Δ_{ik}^j, and the fact that $(R_Q \otimes R_Q) \circ \mathrm{Ad}(\Sigma)$ provides a coinvolution.

We show that it is an isomorphic copy of P. If $z \in \tilde{P}_+$, it is easily seen that $(\psi_N \otimes \iota)(z) \in (O^{\gamma_O})^{+,\mathrm{ext}}$, so since γ_O is ergodic, $\psi_N \otimes \iota$ restricted to $\tilde{P}$ yields an nsf weight $\psi_{\tilde{P}}$ (it is semi-finite since $\tilde{P}$ contains $\beta_P(P)$, on which $\psi_N \otimes \iota$ is semi-finite by right invariance of ψ_N with respect to β_P). By the formula for the comultiplication and the invariance property of ψ_N, it is also immediate that $\psi_{\tilde{P}}$ is a right invariant weight for $\tilde{P}$, hence $\tilde{P}$ is a von Neumann algebraic quantum group. Since the comultiplication $\Delta_{\tilde{P}}$ can be written as $z \to \tilde{W}_P^*(1 \otimes z)\tilde{W}_P$ where $\tilde{W}_P = W^\Theta \otimes 1$ (using that W^Θ implements Θ), since $\{(\iota \otimes \omega)(\Delta_{\tilde{P}}(\tilde{P})) \mid \omega \in \tilde{P}_*\}$ will be σ-weakly dense in

$\tilde{P}$, and since the first leg of $\tilde{W}_P$ lives inside $\beta(P)$, it is clear that $(\tilde{P}, \Delta_{\tilde{P}})$ is just $(\beta_P(P), (\beta_P \otimes \beta_P) \circ \Delta_P \circ \beta_P^{-1})$.

□

Remark: Since $J_{\hat{N}} : \mathscr{L}^2(N) \to \mathscr{L}^2(O)$ is an anti-unitary going *out* of $\mathscr{L}^2(N)$, it contains no information. As already mentioned at some point, this allows us to identify $\mathscr{L}^2(O)$ with the conjugate Hilbert space $\overline{\mathscr{L}^2(N)}$, and then $J_{\hat{N}}$ becomes the canonical anti-unitary conjugation map. This identification precisely induces the identification of O with $\overline{N} = J_{\hat{N}} N J_{\hat{O}}$, the conjugate von Neumann algebra. It is also easy to see that γ_O is then just the left coaction

$$\gamma_O(x) = (R_M \otimes \mathscr{C}^{-1})(\alpha_N^{\mathrm{op}}(\mathscr{C}(x))),$$

where $\mathscr{C}(x) = J_{\hat{O}} x^* J_{\hat{N}}$ for $x \in \overline{N}$. This means that we can construct the von Neumann algebra $\tilde{P}$ rather quickly, just from the coaction α_N. Of course, it takes some more work to show that it has a well-behaved comultiplication (for which we need the Galois unitary), and it would probably take the most work to construct the invariant weights (which we have not tried to obtain in this direct way).

7.5 Comonoidal W*-Morita equivalence

We show that 'being co-monoidally W*-Morita-equivalent' is an equivalence relation. This follows from performing certain operations on linking von Neumann algebraic quantum groupoids.

Reflexivity is clear: $\widehat{M} \otimes M_2(\mathbb{C}) = \begin{pmatrix} \widehat{M} & \widehat{M} \\ \widehat{M} & \widehat{M} \end{pmatrix}$ has an obvious structure of a linking von Neumann algebraic quantum groupoid between $\widehat{M}$ and itself. We call this *the identity linking von Neumann algebraic quantum groupoid.* As for symmetry, note that if $(\widehat{Q}, e)$ is a linking von Neumann algebraic quantum groupoid between $\widehat{M}$ and $\widehat{P}$, then $(\widehat{Q}, 1_{\widehat{Q}} - e)$ is a linking von Neumann algebraic quantum groupoid between $\widehat{P}$ and $\widehat{M}$. We call this *the inverse linking von Neumann algebraic quantum groupoid*.

We now show transitivity. We do this by composing linking von Neumann algebraic quantum groupoids (between), calling the resulting structure the *composite linking von Neumann algebraic quantum groupoid*. Suppose $\widehat{Q}_1$ and $\widehat{Q}_2$ are linking von Neumann algebraic quantum groupoids (between). Consider the associated 3×3 von Neumann linking algebra $\widehat{\mathscr{Q}} = (\widehat{\mathscr{Q}}_{ij})_{i,j\in\{1,2,3\}}$, represented on $\begin{pmatrix} \mathscr{L}^2(\widehat{Q}_{12}) \\ \mathscr{L}^2(\widehat{Q}_{22}) \\ \mathscr{L}^2(\widehat{Q}_{32}) \end{pmatrix}$. Then $\widehat{\mathscr{Q}}_{13}$ is the space of intertwiners between the right representations of $\widehat{Q}_{22}$ on $\mathscr{L}^2(\widehat{Q}_{32})$ and $\mathscr{L}^2(\widehat{Q}_{12})$. This way we can define a map

$$\widehat{\Delta}_{13} : \widehat{\mathscr{Q}}_{13} \to \widehat{\mathscr{Q}}_{13} \otimes \widehat{\mathscr{Q}}_{13} : x \to (\widehat{W}^2_{12})^*(1 \otimes x)\widehat{W}^2_{32},$$

which will be well-defined (by a similar argument as in Lemma 7.3.1) and coassociative. Then we can define

$$\widehat{\Delta}_{31} : \widehat{\mathscr{Q}}_{31} \to \widehat{\mathscr{Q}}_{31} \otimes \widehat{\mathscr{Q}}_{31} : x \to \widehat{\Delta}_{13}(x^*)^*,$$

and then $\widehat{\Delta}_{13}(y)\widehat{\Delta}_{31}(x) = \widehat{\Delta}_{11}(yx)$ and $\widehat{\Delta}_{31}(x)\widehat{\Delta}_{13}(y) = \widehat{\Delta}_{33}(xy)$ for $x \in \widehat{\mathscr{Q}}_{31}, y \in \widehat{\mathscr{Q}}_{13}$. This provides us with a linking von Neumann algebraic quantum groupoid between $\mathscr{Q}_{11} \cong \widehat{Q}_{11} = \widehat{Q}_{1,11}$ and $\mathscr{Q}_{33} \cong \widehat{Q}_{33} = \widehat{Q}_{2,22}$, by which it follows that Q_{11} and Q_{33} are monoidally W*-co-Morita equivalent.

We now present these constructions on the dual level of bi-Galois objects. First of all, if M is a von Neumann algebraic quantum group, (M, Δ_M, Δ_M) is an M-M-bi-Galois object (which we then call *the identity bi-Galois object*). Second, if N is an M-P-bi-Galois object, O will be a P-M-bi-Galois object (which we call *the inverse bi-Galois object*).

To show the transitivity of the monoidal W*-co-Morita relation and its relation with the transitivity of the comonoidal W*-Morita equivalence relation, we need a lemma.

Lemma 7.5.1. *Let N be a right Galois object for a von Neumann algebraic quantum group M, and let $N \subseteq \tilde{N}$ be a unital normal inclusion of von Neumann algebras. Suppose $\alpha_{\tilde{N}}$ is an ergodic right coaction of M on $\tilde{N}$ which restricts to α_N on N. Then $\tilde{N} = N$.*

Proof. It is clear that $\alpha_{\tilde{N}}$ will again be integrable. Since $\varphi_{\tilde{N}} = (\iota \otimes \varphi_M)\alpha_{\tilde{N}}$ restricts to $\varphi_N = (\iota \otimes \varphi_M)\alpha_N$ on N^+, there is a natural isometry v :

$\mathscr{L}^2(N) \to \mathscr{L}^2(\tilde{N})$, sending $\Lambda_{\varphi_N}(x)$ to $\Lambda_{\varphi_{\tilde{N}}}(x)$ for $x \in \mathscr{N}_{\varphi_N}$. Denote $p = vv^*$.

Let $\tilde{G}_{\tilde{N}}$ be the Galois isometry for $\alpha_{\tilde{N}}$. Then we know that $(1 \otimes \theta_{\tilde{N}}(z))\tilde{G}_{\tilde{N}} = \tilde{G}_{\tilde{N}}(1 \otimes \theta_{\tilde{N}}(z))$ for $z \in \tilde{N}$. Since $\tilde{G}_{\tilde{N}}(v \otimes v) = (1 \otimes v)\tilde{G}_N$, where $\tilde{G}_N$ is the Galois unitary for α_N, we see that the range of $\tilde{G}_{\tilde{N}}$ contains the algebraic tensor product $\mathscr{L}^2(M) \odot (\theta_{\tilde{N}}(\tilde{N})v\mathscr{L}^2(N))$. Since this last space has $\mathscr{L}^2(M) \otimes \mathscr{L}^2(\tilde{N})$ as its closure, it follows that $\tilde{G}_{\tilde{N}}$ is unitary, hence $\alpha_{\tilde{N}}$ Galois.

Now
$$\hat{\pi}'_{\alpha_{\tilde{N}}}((\iota \otimes \omega_{\xi,\eta})(V_M))\Lambda_{\varphi_{\tilde{N}}}(x) = \Lambda_{\varphi_{\tilde{N}}}((\iota \otimes \omega_{\delta_M^{1/2}\xi,\eta})(\alpha_N(x)))$$
for $\xi, \eta \in \mathscr{L}^2(M)$ with $\xi \in \mathscr{D}(\delta_M^{1/2})$, and $x \in \mathscr{N}_{\varphi_N}$. Hence $\hat{\pi}'_{\alpha_{\tilde{N}}}(m)v = v\hat{\pi}'_{\alpha_N}(m)$ for $m \in \widehat{M}'$, from which it follows that $p \in \hat{\pi}'_{\alpha_{\tilde{N}}}(\widehat{M}')'$.

So if $\hat{P}$ is the reflected von Neumann algebraic quantum group of $\widehat{M}$ across $\tilde{N}$, it follows that p is a projection in $\hat{P}$ satisfying
$$\begin{aligned} \Delta_{\hat{P}}(p) &= \tilde{G}^*_{\tilde{N}}(1 \otimes vv^*)\tilde{G}_{\tilde{N}} \\ &= (v \otimes v)\tilde{G}^*_N \tilde{G}_N(v \otimes v^*) \\ &= (p \otimes p). \end{aligned}$$
Then necessarily $p = 1$ by Lemma 6.4 of [56], so $\tilde{N} = N$.

□

Suppose now that $(Q_{12}, \gamma_{12}, \alpha_{12})$ and $(Q_{23}, \gamma_{23}, \alpha_{23})$ are respectively Q_{11}-Q_{22} and Q_{22}-Q_{33}-bi-Galois objects for certain von Neumann algebraic quantum groups Q_{ii}. Denote
$$Q_{13} = \{x \in Q_{12} \otimes Q_{23} \mid (\alpha_{12} \otimes \iota)(x) = (\iota \otimes \gamma_{23})(x)\},$$
and let α_{13} be the restriction of $(\iota \otimes \alpha_{23})$ to Q_{13}, and γ_{13} the restriction of $(\gamma_{12} \otimes \iota)$ to Q_{13}. Then $(Q_{13}, \alpha_{13}, \gamma_{13})$ will be a Q_{11}-Q_{33}-bi-Galois object, which we will call *the composite bi-Galois object*. To see this, we show that it is isomorphic to the bi-Galois object associated to the composition of their associated linking von Neumann algebraic quantum groupoids.

For consider again the 3×3 linking von Neumann algebra $\hat{\mathscr{Q}}$ associated to their (dual) linking von Neumann algebras. Then it is easy to check that $((\hat{\mathscr{Q}}_{ij}), (\hat{\Delta}_{ij}))$ has the structure of a measured quantum groupoid with base

$\mathbb{C}^3$ in the obvious way. Then the dual of this '3×3-linking quantum groupoid' can again be written as $\mathcal{Q} = \bigoplus_{i,j=1}^{3} \mathcal{Q}_{ij}$, with the dual comultiplication $\Delta_{\mathcal{Q}}$ splitting up into maps

$$\Delta_{ij}^{k} : \mathcal{Q}_{ij} \to (\mathcal{Q}_{ik} \otimes \mathcal{Q}_{kj}).$$

The triple $(\mathcal{Q}_{13}, \Delta_{13}^{3}, \Delta_{13}^{1})$ will then be the Q_{11}-Q_{33}-bi-Galois object associated with the linking von Neumann algebraic quantum groupoid $\begin{pmatrix} \hat{\mathcal{Q}}_{11} & \hat{\mathcal{Q}}_{13} \\ \hat{\mathcal{Q}}_{31} & \hat{\mathcal{Q}}_{33} \end{pmatrix}$.

We show that this bi-Galois object is isomorphic with $(Q_{13}, \alpha_{13}, \gamma_{13})$. We have that $\Delta_{12}^{1} = \gamma_{12}$, $\Delta_{12}^{2} = \alpha_{12}$, $\Delta_{23}^{2} = \gamma_{23}$ and $\Delta_{23}^{3} = \alpha_{23}$ (identifying $(Q_{ik}, \Delta_{ik}^{j})$ with $(\mathcal{Q}_{ik}, \Delta_{ik}^{j})$ when $|i - k| < 2$), and by coassociativity, it is easily seen that Δ_{13}^{2} sends $\mathcal{Q}_{13}$ into Q_{13}. Moreover, for $x \in \mathcal{Q}_{13}$, we have

$$\alpha_{13}(\Delta_{13}^{2}(x)) = (\Delta_{13}^{2} \otimes \iota)(\Delta_{13}^{3}(x)),$$

and

$$\gamma_{13}(\Delta_{13}^{2}(x)) = (\iota \otimes \Delta_{13}^{2})(\Delta_{13}^{1}(x)).$$

So to end the proof, we have to show that Q_{13} is exactly the image of $\mathcal{Q}_{13}$ under Δ_{13}^{2}. But α_{13} is an ergodic coaction. Since (Q_{13}, α_{13}) contains the Galois object $(\Delta_{13}^{2}(\mathcal{Q}_{13}), \alpha_{13})$, we must have $Q_{13} = \Delta_{13}^{2}(\mathcal{Q}_{13})$ by Lemma 7.5.1.

This provides us then with a canonical composition of two bi-Galois objects of which the first has its right coacting von Neumann algebraic quantum group equal to the left coacting one of the second. Since this composition is easily seen to pass to isomorphism classes, we can make a (large) category containing as objects all von Neumann algebraic quantum groups, and with morphisms isomorphism classes of bi-Galois objects between the corresponding von Neumann algebraic quantum groups, for it is easily seen that the composition will be associative, and that the isomorphism class of (M, Δ_M, Δ_M) for a von Neumann algebraic quantum group M will provide a unit morphism at M. In fact, this will be a (large) *groupoid*: Proposition 7.4.19 shows that both compositions of (N, γ_N, α_N) and (O, γ_O, α_O) will be isomorphic to the identity morphisms. We can interpret this large groupoid as a big '2-cohomology groupoid', jointly for all von Neumann algebraic quantum groups together. We will treat a subgroupoid of it in the ninth chapter (see in particular Proposition 9.1.5 as to why we use the terminology of 2-cohomology).

7.6 C*-algebraic structures

For the rest of this subsection, *let* (N, γ_N, α_N) *be a fixed* P*-*M*-bi-Galois object between certain von Neumann algebraic quantum groups* M *and* P. We will apply to the associated linking von Neumann algebra $(\widehat{Q}, e)$ and its dual Q the C*-algebraic constructions explained in the second and third sections of Chapter 11.

Let A be the reduced C*-algebraic quantum group associated to M, and D the one associated to P. Let A^u be the universal C*-algebraic quantum group associated to M, and D^u the one associated to P. We use the obvious notation for the duals. We also use notation as before for the associated structures.

Theorem 7.6.1. *Let* N *be a* P*-*M*-bi-Galois object. Then the C*-algebras* $\widehat{A}$ *and* $\widehat{C}$ *are C*-Morita equivalent.*

Proof. Denote the weak Hopf C*-algebra associated to $(\widehat{Q}, e)$ by $(\widehat{E}, e)$, where e is interpreted in the obvious way as an element in $M(\widehat{E})$. Then $\widehat{E}$ is the normclosure of the set

$$\{\begin{pmatrix} (\iota \otimes \omega_{11})(\widehat{W}_{11}) & (\iota \otimes \omega_{12})(\widehat{W}_{12}) \\ (\iota \otimes \omega_{21})(\widehat{W}_{21}) & (\iota \otimes \omega_{22})(\widehat{W}_{22}) \end{pmatrix} \mid \omega_{ij} \in (Q_{ij})_*\} \subseteq \widehat{Q}.$$

Then $\widehat{E}$ is a linking C*-algebra, once we have shown that $\widehat{B}\widehat{B}^*$ equals the normclosure of $\{(\omega_{11} \otimes \iota)(W_{11}) \mid \omega_{11} \in (Q_{11})_*\}$. (The density of $\widehat{B}^*\widehat{B}$ in $\widehat{A}$ follows by symmetry.)

Choose $\omega \in B(\mathscr{L}^2(N))_*$ and $\omega' \in B(\mathscr{L}^2(O))_*$, then $(\omega \otimes \iota)(W_{12}^2) \in \widehat{\pi}_{12}^2(\widehat{B})$, $(\omega' \otimes \iota)(W_{21}^2) \in \widehat{\pi}_{12}^2(\widehat{B})^*$. By the pentagon identities in Lemma 7.4.13,

$$\begin{aligned}(\omega \otimes \iota)(W_{12}^2)(\omega' \otimes \iota)(W_{21}^2) &= (\omega \otimes \omega' \otimes \iota)((W_{12}^2)_{13}(W_{21}^2)_{23}) \\ &= (\omega \otimes \omega' \otimes \iota)((W_{12}^1)_{12}^*(W_{11}^2)_{23}(W_{12}^1)_{12}),\end{aligned}$$

from which it follows that $\widehat{\pi}_{11}^2(\widehat{B}\widehat{B}^*)$ is dense in $\widehat{\pi}_{11}^2(\widehat{D})$.

□

Let us now look at the reduced C*-algebra E pertaining to the co-linking von Neumann algebraic quantum groupoid Q. We denote by $[\,\cdot\,]$ the normclosure of the linear span of a set.

Proposition 7.6.2. *Let N be a P-M-bi-Galois object.*

1. *The closure of*
$$\{(\omega \otimes \iota)(\tilde{G}) \mid \omega \in \hat{O}_*\}$$
is a C^*-algebra B.

2. *The restrictions of the coactions α_N and γ_N to B are continuous in the strong sense (cf. section 5 of [5]), and satisfy*
$$[\alpha_N(B)(B \otimes 1)] = B \underset{min}{\otimes} A,$$
$$[(1 \otimes B)\gamma_N(B)] = D \underset{min}{\otimes} B.$$

Proof. For the first statement, note that E is the closure of

$$\{(\omega \otimes \iota)(W_{\hat{Q}}) \mid \omega \in \hat{Q}_*\} = \{\sum_{i,j}^{2}(\omega_{ik} \otimes \iota)(\widehat{W}^2_{ki}) \mid \omega_{ik} \in B(\mathscr{L}^2(N) \oplus \mathscr{L}^2(M))_*\}.$$

This means that E splits into a direct sum $D \oplus C \oplus B \oplus A$, where B is the normclosure of $\{(\omega \otimes \iota)(\widehat{W}^2_{12}) \mid \omega \in B(\mathscr{L}^2(N) \oplus \mathscr{L}^2(M))_*\}$, and $C = J_{\hat{N}} B J_{\hat{O}}$. Since $\widehat{W}^2_{12} = \tilde{G}$, the first result follows.

The second statement follows immediately from the fact that

$$[\Delta_Q(E)(1 \otimes E)] = [\Delta_Q(E)(E \otimes 1)] = \Delta_Q(1)(E \underset{\text{min}}{\otimes} E),$$

which was proven in Proposition 11.2.2.

□

Now we look at the preduals. Give M_* and P_* the *-Banach algebra structure by the usual predual norm, the product $\omega_1 \cdot \omega_2 = (\omega_1 \otimes \omega_2) \circ \Delta$, and, momentarily, with the *-operation *determined by the unitary antipode* (so $\omega^*(x) = \overline{\omega}(R(x))$). In the following proposition, a topologically strict P_*-M_*-imprimitivity bimodule is taken in the sense of Definition XI.7.1 of [36] (see also section 5 of [51]).

Proposition 7.6.3. *Let N be a P-M-bi-Galois object. Then N_* is a topologically strict P_*-M_*-imprimitivity bimodule.*

Proof. We can also give Q_* a Banach *-algebra structure by the usual multiplication, and $\omega \to \overline{\omega} \circ R_Q$ as the *-operation. Then it is clear that inside this algebra, $P_* \cdot N_* \cdot M_* \subseteq N_*$, so that N_* is at least a P_*-M_*-bimodule. Now for $\omega_1, \omega_2 \in N_*$, define $\langle \omega_1, \omega_2 \rangle_M = \omega_1^* \cdot \omega_2$ and $\langle \omega_1, \omega_2 \rangle_P = \omega_1 \cdot \omega_2^*$. Then clearly $\langle \cdot, \cdot \rangle_M$ has range in M_*, $\langle \cdot, \cdot \rangle_P$ has range in P_*, and these make N_* into a P_*-M_*-imprimitivity bimodule.

So we only have to see if this imprimivity bimodule is strict. Suppose $m \in M$ is such that $\langle \omega_1, \omega_2 \rangle_M(m) = 0$ for all $\omega_1, \omega_2 \in N_*$. Then $(\omega_1 \otimes \omega_2)\beta_M(m) = 0$ for all $\omega_1 \in O_*$ and $\omega_2 \in N_*$. Hence $\beta_M(m) = 0$, and $m = 0$.

□

Now we look at the universal level. The Banach *-algebra $\mathscr{L}^1_*(Q)$ consists of those $\omega \in Q_*$ for which $x \in \mathscr{D}(\tau^Q_{-i/2}) \to \overline{\omega}(R_Q(\tau^Q_{-i/2}(x)))$ extends to a normal functional ω^* on Q, with as product the one introduced before on Q_*, with this *new* * as the involution, and with norm the maximum norm of $\| \cdot \|$ and $\| \cdot {}^* \|$. We will use the corresponding notation for M and P. Denote by $\widehat{E}^u$ the universal C*-algebra associated with $(\widehat{Q}, e)$ (as explained in the last section of the chapter 11).

Theorem 7.6.4. *Let N be a P-M-bi-Galois object, and $\widehat{A}^u$ and $\widehat{D}^u$ the universal C*-algebraic quantum groups associated with resp. $\widehat{M}$ and $\widehat{P}$.*

1. *$\mathscr{L}^1_*(Q)$ has the form $\begin{pmatrix} \mathscr{L}^1_*(P) & \mathscr{L}^1_*(N) \\ \mathscr{L}^1_*(O) & \mathscr{L}^1_*(M) \end{pmatrix}$, with $\mathscr{L}^1_*(N)$ a topologically strict $\mathscr{L}^1_*(P)$-$\mathscr{L}^1_*(M)$-imprimitivity bimodule.*
2. *$\widehat{E}^u$ is of the form $\begin{pmatrix} \widehat{D}^u & \widehat{B}^u \\ \widehat{C}^u & \widehat{A}^u \end{pmatrix}$, and then $\widehat{B}^u$ is an $\widehat{A}^u$-$\widehat{D}^u$-equivalence bimodule. In particular, $\widehat{A}^u$ and $\widehat{D}^u$ are C*-Morita equivalent.*

Proof. As a vector space, we have $Q_* = P_* \oplus O_* \oplus N_* \oplus M_*$. Denoting

$$\mathscr{L}^1_*(N) = \{\omega \in N_* \mid \exists \omega_\tau \in N_* : \forall x \in \mathscr{D}(\tau^N_{i/2}) : \omega_\tau(x) = \omega(\tau^N_{i/2}(x))\},$$

and similarly for O, it is then easy to see that also

$$\mathscr{L}^1_*(Q) = \mathscr{L}^1_*(P) \oplus \mathscr{L}^1_*(O) \oplus \mathscr{L}^1_*(N) \oplus \mathscr{L}^1_*(M)$$

as a vector space, since the restriction of the antipode of Q to M and P gives their respective antipodes. Since the multiplication of the components is easily seen to correspond to a matrix multiplication, we get the first part of the first statement.

The fact that $\mathscr{L}^1_*(N)$ is a topologically strict $\mathscr{L}^1_*(P)$-$\mathscr{L}^1_*(M)$-imprimitivity bimodule can be proven exactly as in the previous proposition. The only thing which may not be clear is why this imprimitivity bimodule still has to be topologically strict. But by the fact that there is a generator for the universal representation λ^u of $\mathscr{L}^1_*(M)$, we can identify $(\mathscr{L}^1_*(M))^*$ with a subspace of M (cf. the remark before Lemma 4.1 of [54]). Then the result follows as in the previous proof, since $\mathscr{L}^1_*(N)$ is normdense in N_* (and $\mathscr{L}^1_*(O)$ in O_*).

The second statement follows immediately from the first one.

□

Again, we also have a result on the dual level. Write E^u for the universal C*-algebra of Q.

Proposition 7.6.5. *1. $\mathscr{L}^1_*(\hat{Q})$ is of the form $\mathscr{L}^1_*(\hat{P})\oplus\mathscr{L}^1_*(\hat{N})\oplus\mathscr{L}^1_*(\hat{O})\oplus\mathscr{L}^1_*(\widehat{M})$ with $\mathscr{L}^1_*(\hat{N})$ a Banach *-algebra,*

2. E^u is of the form $D^u\oplus C^u\oplus B^u\oplus A^u$ for certain C-algebras B^u and C^u,*

3. B^u is the universal enveloping C-algebra of $\mathscr{L}^1_*(\hat{O})$.*

Proof. Using notation as in the third part of Chapter 11, define $\mathscr{L}^1_*(\hat{N}) = m^{\hat{d}}_{e_1}\cdot\mathscr{L}^1_*(\hat{Q})\cdot m^d_{e_2}$ and $\mathscr{L}^1_*(\hat{O}) = m^{\hat{d}}_{e_2}\cdot\mathscr{L}^1_*(\hat{Q})\cdot m^d_{e_1}$. Then the first statement is obvious. Defining $B^u = d^u(e_2)E^u\hat{d}^u(e_1)$ and $C^u = d^u(e_1)E^u\hat{d}^u(e_2)$, the second statement is obvious.[3] Also the third statement is immediate.

□

Remark: It is easy to see that if $\hat{A}$ and $\hat{D}$ are two reduced C*-algebraic quantum groups, which are C*-Morita equivalent by a linking C*-algebra with a compatible comultiplication structure, then the associated von Neumann algebraic quantum groups are comonoidally W*-Morita equivalent. This is

[3]The ordering may seem strange, but note that under duality, N_* corresponds to $\hat{N}$, but $\hat{N}_*$ corresponds to O. Indeed: $\widehat{W}_{ij}\in\hat{Q}_{ji}\otimes Q_{ij}$, while $W_{ij} := \Sigma\widehat{W}^*_{ij}\Sigma\in Q_{ij}\otimes\hat{Q}_{ij}$. Hence B^u really corresponds to B, which in turn corresponds to N (by σ-weakly closing B).

no longer clear (to me) when passing to the universal level: for this to be true, one would (only) need to show that the supports of the left invariant weights of the two quantum groups, inside the universal von Neumann algebraic envelope of the linking C*-algebra, are not central.

Chapter 8

Construction methods

In this chapter, we consider the interplay between Galois objects (or coactions) and quantum sub-(or over-)groups.

8.1 Reduction

8.1.1 Restriction of Galois coactions

Lemma 8.1.1. *Let $\widehat{M}_1$ be a closed quantum subgroup of the von Neumann algebraic quantum group $\widehat{M}$, and let α be an integrable right coaction of M on a von Neumann algebra N. Then the restriction α_1 of α to M_1 is again integrable, and $\widehat{\pi}'_{\alpha_1}(x) = \widehat{\pi}'_{\alpha}(x)$ for $x \in \widehat{M}'_1$.*

Proof. First, we claim that there is a *-isomorphism Φ from the crossed product $N \rtimes M_1$ to the sub-von Neumann algebra of $N \rtimes M$ generated by $\alpha(N)$ and $(1 \otimes \widehat{M}'_1)$, sending $\alpha_1(x)$ to $\alpha(x)$ for $x \in N$, and $1 \otimes m$ to $1 \otimes m$ for $m \in \widehat{M}'_1$. Indeed: applying $\alpha \otimes \iota_{B(\mathscr{L}^2(M_1))}$ to $N \rtimes M_1$ and using the definition of restriction, $N \rtimes M_1$ gets sent to the von Neumann algebra generated by $(\iota_N \otimes \alpha_M)\alpha(N)$ and $(1 \otimes 1 \otimes \widehat{M}'_1)$ on $\mathscr{L}^2(N) \otimes \mathscr{L}^2(M) \otimes \mathscr{L}^2(M_1)$, where α_M is the canonical right coaction of M_1 on M. But this is then a sub-von Neumann algebra of $N \otimes (M \rtimes M_1)$. Using the Galois homomorphism for α_M, we can then represent it as a sub-von Neumann algebra of $N \rtimes M$, and it is clear that this will just be the stated sub-von Neumann algebra. So we can take $\Phi = (\iota_N \otimes \rho_{\alpha_M})(\alpha \otimes \iota_{B(\mathscr{L}^2(M_1))})$.

Now by Lemma 6.5.4, we will have that $\widehat{\pi}'_{\alpha}$, which we momentarily view as a right representation of $\widehat{A}^u$, will restrict to $\widehat{\pi}'_{\alpha_1}$ on $\widehat{A}^u_1$. Hence if $\rho_1 = \rho_\alpha \circ \Phi$, we see that it satisfies $\rho_1(\alpha_1(x)) = x$ for $x \in N$ and $\rho_1(1 \otimes (\iota \otimes \omega)(V_{M_1})) =$

$(\iota\otimes\omega)(U_1)$ for $\omega\in(M_1)_*$, where U_1 is the unitary implementation of α_1. By Proposition 5.3 of [85], we conclude that α_1 is integrable, and then the equalities $\widehat{\pi}'_{\alpha_1}(x)=\widehat{\pi}'_{\alpha}(x)$ for $x\in\widehat{M}'_1$ also follow immediately.

□

Proposition 8.1.2. *Let $\widehat{M}$ be a von Neumann algebraic quantum group, and $\widehat{M}_1$ a closed quantum subgroup. Let α be a right Galois coaction of M on a von Neumann algebra N. Then the restriction α_1 of α to M_1 is still Galois.*

Proof. As we have seen in the previous lemma, α_1 is integrable. Furthermore, its Galois homomorphism is a restriction of ρ_α, hence faithful whenever ρ_α is faithful.

□

8.1.2 Reduction of Galois objects

Proposition 8.1.3. *Let M be a von Neumann algebraic quantum group, and M_1 a closed quantum subgroup. Let N be a right M-Galois object. Denote $N_1=\{x\in N\mid\alpha_N(x)\in N\otimes M_1\}$. Then the restriction of α_N to N_1 makes (N_1,α_{N_1}) into a right M_1-Galois object. Moreover, the reflection P_1 of M_1 across N_1 is then a closed quantum subgroup of the reflection P of M across N, in a canonical way.*

Proof. For the right Galois object N, we will use notations as before.

First note that α_{N_1} is a right coaction on N_1: for $x\in N_1$ and $\omega\in M_*$, we have that $\alpha_N((\iota\otimes\omega)\alpha_{N_1}(x))=(\iota\otimes\iota\otimes\omega)((\iota\otimes\Delta_{M_1})\alpha_N(x))\in N\otimes M_1$. Hence $\alpha_{N_1}(N_1)\subseteq N_1\otimes M_1$. Since α_N is a coaction and Δ_M restricts to Δ_{M_1} on M_1, we have that α_{N_1} is a right coaction.

Now denote $O_1=R_Q(N_1)$. Since $\gamma_O\circ R_Q=(R_Q\otimes R_Q)\circ\alpha_N^{\mathrm{op}}$, and $R_M(M_1)=M_1$ ([4], Prop. 10.5), we can also characterize O_1 as

$$O_1=\{z\in O\mid\gamma_O(z)\in M_1\otimes O\}.$$

Now denote

$$\tilde{P}_1=\{z\in N_1\otimes O_1\mid(\alpha_N\otimes\iota_O)(z)=(\iota_N\otimes\gamma_O)(z)\},$$

and denote $P_1=\beta_P^{-1}(\tilde{P}_1)$, so that P_1 is a von Neumann subalgebra of P. Then $\Delta_P(P_1)\subseteq P_1\otimes P_1$. Indeed: applying $\beta_P\otimes\beta_P$ to $\Delta_P(z)$ for

$z \in P_1$, and using that $(\beta_P \otimes \beta_P)\Delta_P = ((\iota_N \otimes \beta_M)\alpha_N \otimes \iota_O)$, we see that $((\beta_P \otimes \beta_P)\Delta_P)(z) \in N_1 \otimes \beta_M(M_1) \otimes O_1$, so we should only check if $\beta_M(M_1) \in O_1 \otimes N_1$. Since $(\iota_O \otimes \alpha_N)\beta_M = (\beta_M \otimes \iota_M)\Delta_M$, and $(\gamma_O \otimes \iota_N)\beta_M = (\iota_M \otimes \beta_M)\Delta_M$, this condition is fulfilled. It is further also easy to check that we have $R_P(P_1) \subseteq P_1$ and $\tau_t^P(P_1) \subseteq P_1$ as well, using the commutations between the Δ_{ij}^k, R_Q and τ_t^Q, and the fact that $R_M(M_1) = M_1$ and $\tau_M(M_1) = M_1$ ([4], Proposition 10.5).

Now using the other direction in Proposition 10.5 of [4], we conclude that (P_1, Δ_{P_1}) is a closed quantum subgroup of (P, Δ_P) (and in particular, is a von Neumann algebraic quantum group).

Now note that α_{N_1} is clearly ergodic. We show that it is integrable. By ergodicity, we have a faithful normal weight $\varphi_{N_1} = (\iota_{N_1} \otimes \varphi_{M_1})\alpha_{N_1}$. Take $m \in \mathscr{M}_{\varphi_{M_1}}^+$ and $\omega \in (O_1)_*^+$. Then by left invariance of φ_{M_1},

$$\begin{aligned} \varphi_{N_1}((\omega \otimes \iota_{N_1})\beta_{M_1}(m)) &= (\iota_{N_1} \otimes \varphi_{M_1})((((\omega \otimes \iota_{N_1})\beta_{M_1}) \otimes \iota_{M_1})\Delta_{M_1}(m)) \\ &= \varphi_{M_1}(m)\omega(1_{O_1}), \end{aligned}$$

so that $(\omega \otimes \iota_{N_1})\beta_{M_1}(m)$ is integrable for φ_{N_1}. From this, the integrability of φ_{N_1} follows.

We now want to show that α_{N_1} is a Galois coaction. We do this by already constructing the associated co-linking von Neumann algebraic quantum groupoid.

Denote $Q_1 = P_1 \oplus O_1 \oplus N_1 \oplus M_1 \subseteq Q$. It is again easy to check that $\Delta_Q(Q_1) \subseteq Q_1 \otimes Q_1$, and that $R_Q(Q_1) \subseteq Q_1$. Denote by Δ_{Q_1} the restriction of Δ_Q to Q_1, and by R_{Q_1} the restriction of R_Q to Q_1. Denote by γ_{N_1} the associated coaction $N_1 \to P_1 \otimes N_1$ of P_1. Then by symmetry, also γ_{N_1} is an ergodic integrable coaction. Denote $\psi_{N_1} = (\psi_{P_1} \otimes \iota)\gamma_{P_1}$, and denote $\varphi_{O_1} = \psi_{N_1} \circ R_{Q_1}$. We want to check that the collection $\varphi_{P_1}, \varphi_{O_1}, \varphi_{N_1}$ and φ_{M_1} satisfies the conditions for left invariant nsf weights on a co-linking von Neumann algebraic quantum groupoid. In fact, apart from trivial cases, symmetry allows us to reduce to two cases, namely the left invariance of the weights with respect to β_{M_1} and γ_{N_1}. For β_{M_1}, the argument has already been given when discussing integrability of α_{N_1}. For γ_{N_1}: choose $\omega \in (P_1)_*^+$, a state $\tilde{\omega}$ on N, and $x \in \mathscr{M}^+_{\varphi_{N_1}}$. Then

$$\begin{aligned}\varphi_{N_1}((\omega \otimes \iota_{N_1})\gamma_{N_1}(x)) &= \varphi_{M_1}((\omega \otimes \tilde{\omega} \otimes \iota_{M_1})((\iota_{N_1} \otimes \alpha_{N_1})\gamma_{N_1}(x))) \\ &= \varphi_{M_1}((((\omega \otimes \tilde{\omega})\gamma_{N_1}) \otimes \iota_{M_1})\alpha_{N_1}(x)) \\ &= ((\omega \otimes \tilde{\omega})\gamma_{N_1})(1_{N_1}) \cdot \varphi_{N_1}(x) \\ &= \omega(1_{N_1}) \cdot \varphi_{N_1}(x).\end{aligned}$$

Since R_{Q_1} is an anti-multiplicative *-involution flipping the comultiplication, Q_1 has the structure of a co-linking von Neumann algebraic quantum groupoid. But then the map

$$\mathscr{L}^2(N_1) \otimes \mathscr{L}^2(N_1) \to \mathscr{L}^2(N_1) \otimes \mathscr{L}^2(M_1) :$$

$$\Lambda_{N_1}(x) \otimes \Lambda_{N_1}(y) \to \Sigma(\Lambda_{N_1} \otimes \Lambda_{M_1})(\alpha_{N_1}(x)(y \otimes 1))$$

will be unitary, since it coincides with a unitary part of the multiplicative partial isometry of Q_1. Hence α_{N_1} is a Galois coaction.

Since $\beta_{P_1}(P_1) = \{z \in N_1 \otimes O_1 \mid (\alpha_{N_1} \otimes \iota_{O_1})(z) = (\iota_{N_1} \otimes \gamma_{O_1})(z)\}$ by construction, with $(\beta_{P_1} \otimes \beta_{P_1})\Delta_{P_1} = ((\iota_{N_1} \otimes \beta_{M_1})\alpha_{N_1} \otimes \iota_{O_1})$, we can canonically identify P_1, as a von Neumann algebraic quantum group, with the reflection of M_1 across N_1, by Proposition 7.4.19. This concludes the proof.

□

Definition 8.1.4. *In the situation of the previous proposition, we call the right M_1-Galois object N_1* the reduction of N to M_1, *and we denote*

$$(N_1, \alpha_{N_1}) = (Red_{M_1}(N), Red_{M_1}(\alpha_N)).$$

8.2 Induction

8.2.1 Induction along Galois objects

In this subsection, given a P-M-bi-Galois object N and a left coaction Υ of M on some von Neumann algebra, we induce it to a left coaction of P (on a possibly different von Neumann algebra). This generalizes Proposition 7.7 of [27]. We then show that this correspondence preserves certain properties of Υ.

So let N be a P-M-bi-Galois object. Suppose Y is a von Neumann algebra, and Υ a left coaction of M on Y. Denote by $\mathrm{Ind}_N(Y) = Y_N$ the von Neumann algebra

$$Y_N := \{x \in N \otimes Y \mid (\alpha_N \otimes \iota_Y)x = (\iota_N \otimes \Upsilon)x\},$$

and by $\mathrm{Ind}_N(\Upsilon) = \Upsilon_N$ the map $(\gamma_N \otimes \iota_Y)_{|Y_N}$. Then since α_N and γ_N commute, it is easily seen that Υ_N has range in $P \otimes Y_N$, and that then Υ_N is a coaction of P on Y_N.

Definition 8.2.1. *In the foregoing situation, we call* (Y_N, Υ_N) *the* induction *of* Υ *(from* M*) along* N *(to* P*).*

Theorem 8.2.2. *The functor* $(Y, \Upsilon) \to ((Y_N)_O, (\Upsilon_N)_O)$ *is naturally equivalent with the identity.*

Remark: We assume that this takes place in the category of left coactions for M, where a morphism is (for example) a unital normal complete contraction between the spaces acted on, intertwining the coaction.

Proof. Consider the map

$$Y \to O \otimes N \otimes Y : x \to (\beta_M \otimes \iota)\Upsilon(x).$$

Then for $x \in Y$, we have

$$\begin{aligned}(\iota_O \otimes \alpha_N \otimes \iota_Y)(\beta_M \otimes \iota_Y)\Upsilon(x) &= (\beta_M \otimes \iota_M \otimes \iota_Y)((\Delta_M \otimes \iota_Y)\Upsilon(x)) \\ &= (\beta_M \otimes \iota_M \otimes \iota_Y)((\iota_M \otimes \Upsilon)\Upsilon(x)) \\ &= (\iota_O \otimes \iota_N \otimes \Upsilon)((\beta_M \otimes \iota_Y)\Upsilon(x)),\end{aligned}$$

and since $(\alpha_O \otimes \iota_N)\beta_M = (\iota_O \otimes \gamma_N)\beta_M$, also

$$(\alpha_O \otimes \iota_N \otimes \iota_Y)((\beta_M \otimes \iota_Y)\Upsilon(x)) = (\iota_O \otimes \gamma_N \otimes \iota_Y)((\beta_M \otimes \iota_Y)\Upsilon(x)),$$

which shows that the given map has range in $(Y_N)_O$.

Now choose $x \in (Y_N)_O$. Then the fact that

$$(\alpha_O \otimes \iota_N \otimes \iota_Y)(x) = (\iota_O \otimes \gamma_N \otimes \iota_Y)(x)$$

implies that $x = (\beta_M \otimes \iota_Y)(z)$ for some $z \in M \otimes Y$, by (a symmetric version of) Proposition 7.4.19. But also

$$\begin{aligned}(\beta_M \otimes \iota_M \otimes \iota_Y)((\Delta_M \otimes \iota_Y)(z)) &= (\iota_O \otimes \alpha_N \otimes \iota_Y)((\beta_M \otimes \iota_Y)(z)) \\ &= (\iota_O \otimes \alpha_N \otimes \iota_Y)(x) \\ &= (\iota_O \otimes \iota_N \otimes \Upsilon)(x) \\ &= (\iota_O \otimes \iota_N \otimes \Upsilon)((\beta_M \otimes \iota_Y)(z)) \\ &= (\beta_M \otimes \iota_M \otimes \iota_Y)((\iota_M \otimes \Upsilon)(z)),\end{aligned}$$

so by injectivity of β_M, we have $(\Delta_M \otimes \iota_Y)(z) = (\iota_M \otimes \Upsilon)(z)$, and by the biduality theorem, Theorem 2.7. of [85], we have $z = \Upsilon(y)$ for some $y \in Y$. Hence the considered map is a bijection.

Finally, we have for any $x \in Y$ that

$$\begin{aligned}(\iota_M \otimes \beta_M \otimes \iota_Y)((\iota_M \otimes \Upsilon)\Upsilon(x)) &= (\iota_M \otimes \beta_M \otimes \iota_Y)((\Delta_M \otimes \iota_Y)\Upsilon(x)) \\ &= (\gamma_O \otimes \iota_N \otimes \iota_Y)((\beta_M \otimes \iota_Y)\Upsilon(x)),\end{aligned}$$

which shows that the map intertwines the coactions of M.

□

Proposition 8.2.3. *Let N be a P-M-bi-Galois object, Υ a left coaction of M on a von Neumann algebra Y, and (Y_N, Υ_N) the induction of Υ along N. Then*

1. *the coaction Υ is ergodic iff the coaction Υ_N is ergodic.*
2. *the coaction Υ is integrable iff the coaction Υ_N is integrable.*
3. *the coaction Υ is Galois iff the coaction Υ_N is Galois.*

Proof. By biduality, we only have to prove the 'only if' statements.

It is easy to show that ergodic coactions get transformed into ergodic coactions: if $x \in Y_N \subseteq N \otimes Y$ is a coinvariant element, then $x = 1 \otimes z$ with $z \in Y$ by the ergodicity of γ. But by the defining property of Y_N, also z is coinvariant for Υ. Hence $Y_N^{\Upsilon_N} = 1_N \otimes Y^{\Upsilon}$, and in particular, Υ_N is ergodic when Υ is.

Now we prove the second point. Suppose that Υ is integrable. Choose $x \in Y^+$ integrable for Υ, and choose $\xi \in \mathscr{L}^2(O)$ with $\xi \in \mathscr{D}(\delta_O^{-1/2})$. Put $\omega = \omega_{\xi,\xi} \in O_*$, and put

$$y = (\omega \otimes \iota_N \otimes \iota_Y)((\beta_M \otimes \iota_Y)\Upsilon(x)).$$

Then y will be an integrable element for Υ_N: to see this, first note that y will be in Y_N by the proof of the proof of the previous theorem. Next,

$$\Upsilon_N(y) = (\omega \otimes \iota_P \otimes \iota_N \otimes \iota_Y)((\alpha_O \otimes \iota_N \otimes \iota_Y)(\beta_M \otimes \iota_Y)\Upsilon(x)).$$

Choose $\omega' \in (N \otimes Y)_*^+$. Put $z = (\iota_O \otimes \omega')((\beta_M \otimes \iota_Y)\Upsilon(x))$. Then

$$(\iota_O \otimes \omega')(\Upsilon_N(y)) = (\omega \otimes \iota_P)\alpha_O(z).$$

Now by the strong form of right-invariance of ψ_O (cf. Lemma 11.1.8), we have that z will be integrable for ψ_O, with

$$\psi_O(z) = \omega'(1_N \otimes (\psi_M \otimes \iota_Y)(\Upsilon(x))).$$

By (a right analogue of) Proposition 4.5 of [30], we conclude that

$$\psi_P((\omega \otimes \iota_P)\alpha_O(z)) = \|\delta_O^{-1/2}\xi\|^2 \cdot \psi_O(z).$$

Hence y is integrable, with

$$(\psi_P \otimes \iota_{Y_N})(\Upsilon_N(y)) = \|\delta_O^{-1/2}\xi\| \cdot (1 \otimes ((\psi_M \otimes \iota_Y)\Upsilon(x))).$$

So to show that Υ_N is integrable, the only thing left to show is that the y of the above form have σ-weakly dense span in Y_N. Now clearly, the σ-weakly dense span of such y contains each element of the form $((\omega\otimes\iota_N)\beta_M\otimes\iota_Y)\Upsilon(x)$ with $x \in Y$ and $\omega \in O_*$. By the biduality property in the previous theorem, it is as well sufficient to show that the linear span of elements of the form $(\omega \otimes \iota_Y)(x)$ with $\omega \in N_*$ and $x \in Y_N$ is σ-weakly dense in Y. But this space contains all elements of the form $((\omega' \otimes \omega)\beta_M \otimes \iota_Y)(\Upsilon(x))$, with $\omega' \in O_*, \omega \in N_*$ and $x \in Y$. By Proposition 7.6.3, it also contains all elements of the form $(\omega \otimes \iota_Y)(\Upsilon(x))$, with $\omega \in M_*$. But this space is known to be σ-weakly dense in Y (see for example the proof of 7.2.6).

Now we prove the third point. Suppose first that Υ is just an integrable left coaction. Let T_Υ be the operator valued weight $Y \to Y^\Upsilon$ associated with Υ, and let $\psi_Y = \mu \circ T_\Upsilon$ with μ an arbitrary nsf weight on Y^Υ. Also, one has that the operator valued weight $(\psi_N \otimes \iota)$ from $(N \otimes Y)^+$ to $(1 \otimes Y)^{+,\mathrm{ext}}$ restricts to the operator valued weight $T_{\Upsilon_N} = (\psi_P \otimes \iota)\Upsilon_N$ from Y_N^+ to $1 \otimes (Y^{\Upsilon})^+$. Applying Lemma 5.7.9, we see that, if $(1 \otimes Y^{\Upsilon}) = Y_N^{\Upsilon_N} \subseteq Y_N \subseteq (Y_N)_2$ is the basic construction, we can realize $(Y_N)_2$ on $\mathscr{L}^2(N) \otimes \mathscr{L}^2(M)$ as the von Neumann algebra generated by operators of the form xy^*, with x and y of the form

$$\Lambda_{\psi_N \otimes \iota}(z) : \mathscr{L}^2(Y) \to \mathscr{L}^2(N) \otimes \mathscr{L}^2(Y) :$$
$$\Lambda_{\psi_Y}(v) \to (\Lambda_{\psi_N} \otimes \Lambda_{\psi_Y})(z(1 \otimes v))$$

for $v \in \mathscr{N}_{\psi_Y}$ and $z \in \mathscr{N}_{T_{\Upsilon_N}}$.

We can also make a *faithful* copy of $N \ltimes Y_N$ on $\mathscr{L}^2(N) \otimes \mathscr{L}^2(Y)$, similar to the construction in Lemma 6.5.6. Namely, we have $P \ltimes Y_N \subseteq (P \ltimes N) \otimes Y$, and we know that the first factor of this tensor product is representable on $\mathscr{L}^2(N)$ in a standard way. Denote this representation of $P \ltimes Y_N$ by $\tilde{F}$. We want to show that for $z \in P \ltimes Y_N$, we have $\tilde{F}(z)F(1_{(Y_N)_2}) = F(\rho_{\Upsilon_N}(z))$.

We first characterize the subspace of $\mathscr{L}^2(N) \otimes \mathscr{L}^2(Y)$ corresponding to the projection $p = F(1_{(Y_N)_2})$.

Take $u \in \mathscr{T}_{\psi_N} \cdot \mathscr{T}_{\psi_N}$. Then there exists a unique normal functional ω_u on N such that $\omega_u(x^*) = \psi_N(x^* u)$ for $x \in \mathscr{N}_{\psi_N}$. Since for $z \in \mathscr{N}_{T_{\Upsilon_N}}$ and arbitrary $\omega \in Y_*$, we have $(\iota \otimes \omega)(z) \in \mathscr{N}_{\psi_N}$ by a Cauchy-Schwarz type inequality, we deduce that $(\psi_N \otimes \iota)(z^*(u \otimes 1)) = (\omega_u \otimes \iota_Y)(z)$. Then for such u and z, and $v \in \mathscr{N}_{\psi_Y}$, we have

$$\begin{aligned} \Lambda_{(\psi_N \otimes \iota)}(z)^*(\Lambda_{\psi_N}(u) \otimes \Lambda_{\psi_Y}(v)) &= \Lambda_{\psi_Y}((\psi_N \otimes \iota)(z^*(u \otimes v))) \\ &= (\omega_u \otimes \iota)(z^*)\Lambda_{\psi_Y}(v). \end{aligned}$$

Now such ω_u are normdense in N_*, and such z are σ-weakly dense in Y_N. Furthermore, we have that

$$K := \{(\omega \otimes \iota)(z) \mid z \in Y_N, \omega \in N_*\}$$

is σ-weakly dense in Y, which was proven while dealing with the second point of the proposition. Hence

$$\Lambda_{(\psi_N \otimes \iota_Y)}(\mathscr{N}_{T_{\Upsilon_N}})^* \cdot (\mathscr{L}^2(N) \otimes \mathscr{L}^2(Y)) \subseteq \mathscr{L}^2(Y)$$

is a dense subspace.

This means that $p(\mathscr{L}^2(N)\otimes\mathscr{L}^2(Y))$ will be the closure of $\Lambda_{\psi_N\otimes\iota}(\mathscr{N}_{T_{\Upsilon_N}})\cdot(\mathscr{L}^2(N)\otimes\mathscr{L}^2(Y))$.

Now take $x\in Y_N$, $y\in\mathscr{N}_{T_{\Upsilon_N}}$ and $z\in\mathscr{N}_{\psi_Y}$. Then we have

$$\begin{aligned} F(x)\Lambda_{\psi_N\otimes\iota}(y)\Lambda_{\psi_Y}(z) &= \Lambda_{\psi_N\otimes\iota}(xy)\Lambda_{\psi_Y}(z) \\ &= (\Lambda_{\psi_N}\otimes\Lambda_{\psi_Y})(xy(1\otimes z)) \\ &= x\Lambda_{\psi_N\otimes\iota}(y)\Lambda_{\psi_Y}(z) \\ &= \tilde{F}(\Upsilon_N(x))\Lambda_{\psi_N\otimes\iota}(y)\Lambda_{\psi_Y}(z), \end{aligned}$$

which shows that $\tilde{F}(\Upsilon_N(x))p=F(x)$ for $x\in Y_N$.

Now choose $\eta\in\mathscr{L}^2(P)$ and $\xi\in\mathscr{D}(\delta_P^{1/2})$, and put $\omega=\omega_{\xi,\eta}$ and $\omega_\delta=\omega_{\delta_P^{1/2}\xi,\eta}$. Let U_{Y_N} be the unitary implementation of Υ_N, and denote $\psi_{Y_N}=\mu\circ T_{\Upsilon_N}$ (where we identify Y^Υ with $1_N\otimes Y^\Upsilon=Y^{\Upsilon_N}$). Choose $x\in\mathscr{N}_{T_{\Upsilon_N}}$. Then for $y\in\mathscr{N}_\mu$, we have

$$\begin{aligned} (\omega\otimes\iota)(U_{Y_N})\Lambda_{T_{\Upsilon_N}}(x)\Lambda_\mu(y) &= (\omega\otimes\iota)(U_{Y_N})\Lambda_{\psi_{Y_N}}(xy) \\ &= \Lambda_{\psi_{Y_N}}(((\omega_\delta\otimes\iota)\Upsilon_N(x))y), \end{aligned}$$

by Definition-Proposition 6.3.11. Hence $(\omega_\delta\otimes\iota)\Upsilon_N(x)\in\mathscr{N}_{T_{\Upsilon_N}}$ with

$$\Lambda_{T_{\Upsilon_N}}((\omega_\delta\otimes\iota)\Upsilon_N(x))=(\omega\otimes\iota)(U_{Y_N})\Lambda_{T_{\Upsilon_N}}(x),$$

by Lemma 5.7.8. Applying F, we get that

$$\Lambda_{(\psi_N\otimes\iota)}((\omega_\delta\otimes\iota)\Upsilon_N(x))=F((\omega\otimes\iota)(U_{Y_N}))\Lambda_{(\psi_N\otimes\iota)}(x).$$

Applying the left hand side to $\Lambda_{\psi_Y}(y)$ with $y\in\mathscr{N}_{\psi_Y}$, we get

$$\begin{aligned} \Lambda_{(\psi_N\otimes\iota)}((\omega_\delta\otimes\iota)\Upsilon_N(x))\Lambda_{\psi_Y}(y) &= (\Lambda_{\psi_N}\otimes\Lambda_{\psi_Y})(((\omega_\delta\otimes\iota)\Upsilon_N(x))(1\otimes y)) \\ &= (\Lambda_{\psi_N}\otimes\Lambda_{\psi_Y})((((\omega_\delta\otimes\iota)\gamma_N)\otimes\iota)(x(1\otimes y))) \\ &= ((\omega\otimes\iota)(U_P)\otimes 1)(\Lambda_{\psi_N}\otimes\Lambda_{\psi_Y})(x(1\otimes y)), \end{aligned}$$

with U_P the unitary corepresentation belonging to γ_N, again by Definition-Proposition 6.3.11. Since $(\omega\otimes\iota)(U_P)\otimes 1=\tilde{F}((\omega\otimes\iota)(W_P^*)\otimes 1)$, we arrive at

$$F((\omega\otimes\iota)(U_{Y_N}))\Lambda_{(\psi_N\otimes\iota)}(x)=\tilde{F}((\omega\otimes\iota)(W_P)\otimes 1)\Lambda_{(\psi_N\otimes\iota)}(x)$$

for $x\in\mathscr{N}_{T_{\Upsilon_N}}$. Hence for $z\in\hat{P}$, we have $F(\hat{\pi}_{\Upsilon_N}(z))=\tilde{F}(z\otimes 1)p$.

From these two calculations, it follows that $\tilde{F}(z)F(1_{(Y_N)_2}) = F(\rho_{\Upsilon_N}(z))$.

Now suppose that Υ is Galois. Then to finish the proof, we only have to show that $p = 1$. But take $y \in \mathscr{N}_{\psi_Y}$. Choose $x \in Y^+$ square integrable for Υ, and choose $\xi \in \mathscr{L}^2(O)$ with $\xi \in \mathscr{D}(\delta_O^{-1/2})$; put $\omega = \omega_{\xi,\xi} \in O_*$, and put

$$z = (\omega \otimes \iota_N \otimes \iota_Y)((\beta_M \otimes \iota_Y)\Upsilon(x)).$$

Then by the proof of the second point (and a Cauchy-Schwartz type inequality), we know that z is square integrable for Υ_N. We can write

$$\begin{aligned}\Lambda_{\psi_N\otimes\iota_Y}(z)\Lambda_{\psi_Y}(y) &= (\Lambda_{\psi_N} \otimes \Lambda_{\psi_Y})((\omega \otimes \iota_N \otimes \iota_Y)((\beta_M \otimes \iota_Y)\Upsilon(x))(1 \otimes y)) \\ &= ((\omega_{\delta_O^{-1/2}\xi,\xi} \otimes \iota)((W_{21}^2)^*) \otimes 1)(\Lambda_{\psi_M} \otimes \Lambda_{\psi_Y})(\Upsilon(x)(1 \otimes y)),\end{aligned}$$

the last step following by Proposition 4.5 of [30]. Since the second leg of $(W_{21}^2)^*$ is σ-weakly dense in $\hat{Q}_{12}$, and since elements of the form $(\Lambda_{\psi_M} \otimes \Lambda_{\psi_Y})(\Upsilon(x)(1 \otimes y))$ are dense in $\mathscr{L}^2(M) \otimes \mathscr{L}^2(Y)$, by the assumption that Υ is Galois, we see that the linear span of elements of the form $\Lambda_{\psi_N\otimes\iota_Y}(z)\eta$, with $z \in \mathscr{N}_{T_{\Upsilon_N}}$ and $\eta \in \mathscr{L}^2(Y)$, is closed in $\mathscr{L}^2(N)\otimes\mathscr{L}^2(Y)$. So we are done.

□

8.2.2 Induction of Galois objects

Proposition 8.2.4. *Let $\widehat{M}_1$ be a closed quantum subgroup of the von Neumann algebraic quantum group $\widehat{M}$. Let (N_1, α_{N_1}) be a right Galois object for M_1. Then the induced coaction $\alpha_N := Ind_M(\alpha_{N_1})$ of α_{N_1} by M makes $N := Ind_M(N_1)$ a right M-Galois object. Moreover, if $\widehat{P}_1$ is the reflection of $\widehat{M}_1$ across N_1, and $\widehat{P}$ the reflection of $\widehat{M}$ across N, then $\widehat{P}_1$ is a closed quantum subgroup of $\widehat{P}$, in a canonical way.*

Proof. We recall that $N = \{z \in N_1 \otimes M \mid (\alpha_{N_1} \otimes \iota_M)z = (\iota_{N_1} \otimes \gamma_M)z\}$, where γ_M is the canonical left coaction of M_1 on M, and that α_N is the restriction of $\iota_{N_1} \otimes \Delta_M$ to N.

First, we show that α_N is ergodic. Suppose $z \in N$ and $\alpha_M(z) = z \otimes 1_M$. Then $(\iota_{N_1}\otimes\Delta_M)(z) = z\otimes 1_M$, so $z = x\otimes 1_M$ with $x \in N_1$. Since $x\otimes 1_{N_1} \in N$, we get $\alpha_{N_1}(x) = x \otimes 1_{M_1}$. So x is scalar by ergodicity of α_{N_1}, and hence z is scalar.

We show integrability of α_N. Choose $m \in \mathscr{M}^+_{\varphi_M}$. Choose $\omega \in (O_1)^+_*$, and denote

$$z = (\omega \otimes \iota_{N_1} \otimes \iota_M)((\beta_{N_1} \otimes \iota_M)\gamma_M)(m) \in N_1 \otimes M.$$

Then $z \in N$: we have

$$\begin{aligned}(\alpha_{N_1} \otimes \iota_M)(z) &= (\omega \otimes \iota_{N_1} \otimes \iota_{M_1} \otimes \iota_M)((\iota_{O_1} \otimes \alpha_{N_1} \otimes \iota_M)(\beta_{M_1} \otimes \iota_M)\gamma_M)(m) \\ &= (\omega \otimes \iota_{N_1} \otimes \iota_{M_1} \otimes \iota_M)((\beta_{M_1} \otimes \iota_{M_1} \otimes \iota_M)(\Delta_{M_1} \otimes \iota_M)\gamma_M)(m) \\ &= (\omega \otimes \iota_{N_1} \otimes \iota_{M_1} \otimes \iota_M)((\beta_{M_1} \otimes \iota_{M_1} \otimes \iota_M)(\iota_{M_1} \otimes \gamma_M)\gamma_M)(m) \\ &= (\iota_{N_1} \otimes \gamma_M)(z).\end{aligned}$$

Furthermore, z will be integrable:

$$\begin{aligned}(\iota_{N_1} \otimes \varphi_M)(\alpha_N(z)) &= (\iota_{N_1} \otimes \varphi_M)(z) \\ &= (\omega \otimes \iota_M)(\beta_{N_1}((\iota_{M_1} \otimes \varphi_M)(\gamma_M(m)))) \\ &= \omega(1_{O_1})\varphi_M(m),\end{aligned}$$

where $(\iota_{M_1} \otimes \varphi_M)(\gamma_M(m)) = \varphi_M(m)$ follows from the fact that $(\iota \otimes \Delta)\Gamma_l = (\Gamma_l \otimes \iota)\Delta$ (cf. Proposition 12.2 of [54]).

Finally, we show that the coaction is Galois. For this, it is enough to show that the canonical map $\rho_{\alpha_N} : N \rtimes M \to B(\mathscr{L}^2(N))$ is injective, by the results of the second section. But by Lemma 6.5.6, $N \rtimes M$ is a type I factor, so that this must be necessarily so.

We now prove the second part of the proposition, concerning the relation between $\hat{P}_1$ and $\hat{P}$. First note that $\hat{Q}_1$ can be represented on

$$\begin{pmatrix} \mathscr{L}^2(\hat{N}_1) \\ \mathscr{L}^2(\widehat{M}_1) \end{pmatrix} \underset{\varphi_{\widehat{M}_1}}{\otimes} \mathscr{L}^2(\widehat{M})$$

by the map $\pi^{\text{ind}} := \pi^{\hat{Q}_1,2} \underset{\varphi_{\widehat{M}_1}}{\otimes} \iota$. In particular, $\hat{O}_1$ is then represented by operators

$$\pi^{\text{ind}}(\hat{O}_1) : \mathscr{H} := \mathscr{L}^2(\hat{N}_1) \underset{\varphi_{\widehat{M}_1}}{\otimes} \mathscr{L}^2(\widehat{M}) \to \mathscr{L}^2(\widehat{M}) \cong \mathscr{L}^2(\widehat{M}_1) \underset{\varphi_{\widehat{M}_1}}{\otimes} \mathscr{L}^2(\widehat{M}).$$

Since the Galois unitary $\tilde{G}_{N_1}$ for α_{N_1} lies in $\hat{O}_1 \otimes N_1$, we can thus form the operator $(\pi^{\text{ind}} \otimes \iota)(\tilde{G}_{N_1})$. Put

$$\tilde{G}_{\text{ind}} := (W_{\widehat{M}})_{13}((\pi^{\text{ind}} \otimes \iota)(\tilde{G}_{N_1}))_{12},$$

which is an operator

$$\tilde{G}_{\mathrm{ind}} : \mathscr{H} \otimes \mathscr{L}^2(N_1) \otimes \mathscr{L}^2(M) \to \mathscr{L}^2(\widehat{M}) \otimes \mathscr{L}^2(N_1) \otimes \mathscr{L}^2(M).$$

Clearly, $\tilde{G}_{\mathrm{ind}} \in B(\mathscr{H}, \mathscr{L}^2(\widehat{M})) \otimes N_1 \otimes M$, and furthermore,

$$\begin{aligned}
&(\iota \otimes \iota \otimes \gamma_M)(\tilde{G}_{\mathrm{ind}}) \\
&\quad = (W_{\widehat{M}})_{14}((\pi^{\mathrm{ind}} \otimes \iota)(W_{\widehat{M}_1}))_{13}((\pi^{\mathrm{ind}} \otimes \iota)(\tilde{G}_{N_1}))_{12} \\
&\quad = (W_{\widehat{M}})_{14}(((\pi^{\mathrm{ind}} \otimes \iota \otimes \iota)((W_{\widehat{M}_1})_{13}(\tilde{G}_{N_1})_{12})) \otimes 1) \\
&\quad = (\iota \otimes \alpha_{N_1} \otimes \iota)(\tilde{G}_{\mathrm{ind}}).
\end{aligned}$$

Hence $\tilde{G}_{\mathrm{ind}} \in B(\mathscr{H}, \mathscr{L}^2(\widehat{M})) \otimes N$. Moreover, it is easily seen that

$$(\iota \otimes \alpha_N)(\tilde{G}_{\mathrm{ind}}) = (W_{\widehat{M}})_{13}(\tilde{G}_{\mathrm{ind}})_{12}.$$

Hence, if $\tilde{G}_N$ denotes the Galois unitary of (N, α_N), with the second leg identified with operators on $\mathscr{L}^2(N_1) \otimes \mathscr{L}^2(M)$, we have $(\iota \otimes \alpha_N)(\tilde{G}_N^* \tilde{G}_{\mathrm{ind}}) = \tilde{G}_N^* \tilde{G}_{\mathrm{ind}} \otimes 1_M$, and thus $\tilde{G}_{\mathrm{ind}} = \tilde{G}_N(u \otimes 1_N)$ for some unitary $u : \mathscr{H} \to \mathscr{L}^2(N)$. Since u is clearly right $\widehat{M}$-linear, we obtain an embedding $\widehat{N}_1 \to \widehat{N} : x \to u\pi^{\mathrm{ind}}(x)$, which can then be extended to a unital normal embedding $F : \widehat{Q}_1 \to \widehat{Q}$.

In particular, we have a unital normal embedding $\widehat{P}_1 \to \widehat{P}$. So to see if this makes $\widehat{P}_1$ a closed quantum subgroup of $\widehat{P}$, we should show that the embedding F intertwines $\Delta_{\widehat{Q}_1}$ and $\Delta_{\widehat{Q}}$. Clearly, it is already sufficient to check this on $\widehat{N}_1$. Now

$$(u \otimes 1_{N_1} \otimes 1_M) \cdot ((\pi^{\mathrm{ind}} \otimes \iota)(\tilde{G}_{N_1})^*_{12}) = \tilde{G}_N^*(W_{\widehat{M}})_{13},$$

by definition of u. Since the first leg of $\tilde{G}_{N_1}^*$ is σ-weakly dense in $\widehat{N}_1$, and $(\Delta_{\widehat{N}_1} \otimes \iota)(\tilde{G}_{N_1}^*) = (\tilde{G}_{N_1})^*_{23}(\tilde{G}_{N_1})^*_{13}$, we have to see if

$$(\Delta_{\widehat{N}} \otimes \iota)(\tilde{G}_N^*(W_{\widehat{M}})_{13}) = (\tilde{G}_N^*)_{234}(W_{\widehat{M}})_{24}(\tilde{G}_N^*)_{134}(W_{\widehat{M}})_{14}.$$

Now the left hand side equals $(\tilde{G}_N^*)_{234}(\tilde{G}_N^*)_{134}(W_{\widehat{M}})_{14}(W_{\widehat{M}})_{24}$. So, we should check if

$$(\tilde{G}_N^*)_{134}(W_{\widehat{M}})_{14}(W_{\widehat{M}})_{24} = (W_{\widehat{M}})_{24}(\tilde{G}_N^*)_{134}(W_{\widehat{M}})_{14}. \tag{8.1}$$

Now we use that $(\iota \otimes \alpha_N)(\tilde{G}_N) = (W_{\widehat{M}})_{13}(\tilde{G}_N)_{12}$ (with the second leg living on $\mathscr{L}^2(N_1) \otimes \mathscr{L}^2(M)$). Since α_N is Δ_M applied to the second leg of an element of N, we deduce from this that $(W_M)^*_{34}(\tilde{G}_N)_{124}(W_M)_{34} = (W_{\widehat{M}})_{14}(\tilde{G}_N)_{123}$ (the middle legs living on respectively $\mathscr{L}^2(N_1)$ and $\mathscr{L}^2(M)$). Rearranging indices, this becomes

$$(W_{\widehat{M}})_{24}(\tilde{G}_N)_{132}(W_{\widehat{M}})^*_{24} = (W_{\widehat{M}})_{12}(\tilde{G}_N)_{134}.$$

Using this equality, the left hand side of (8.1) can be rewritten as follows:

$$\begin{aligned}
&(\tilde{G}^*_N)_{134}(W_{\widehat{M}})_{14}(W_{\widehat{M}})_{24} \\
&= (\tilde{G}^*_N)_{134}(W_{\widehat{M}})^*_{12}(W_{\widehat{M}})_{24}(W_{\widehat{M}})_{12} \\
&= ((W_{\widehat{M}})_{24}(\tilde{G}_N)^*_{132}(W_{\widehat{M}})^*_{24})((W_{\widehat{M}})_{24}(W_{\widehat{M}})_{12}) \\
&= (W_{\widehat{M}})_{24}(\tilde{G}_N)^*_{132}(W_{\widehat{M}})_{12}.
\end{aligned}$$

So the identity (8.1) is proven if we can show that

$$(W_{\widehat{M}})_{24}(\tilde{G}_N)^*_{132}(W_{\widehat{M}})_{12} = (W_{\widehat{M}})_{24}(\tilde{G}^*_N)_{134}(W_{\widehat{M}})_{14}. \tag{8.2}$$

After canceling and rearranging indices, the identity (8.2) becomes

$$(\tilde{G}_N)^*_{123}(W_{\widehat{M}})_{13} = (\tilde{G}_N)^*_{124}(W_{\widehat{M}})_{14}.$$

But both sides equal $(u \otimes 1_{N_1} \otimes 1_M \otimes 1_M) \cdot ((\pi^{\text{ind}} \otimes \iota)(\tilde{G}_{N_1})^*)_{12}$. So we are done.

□

Lemma 8.2.5. *Let $\widehat{M_1}$ be a closed quantum subgroup of the von Neumann algebraic quantum group $\widehat{M}$. Let (N_1, α_{N_1}) be a right Galois object for M_1. Then $Ind_M(N_1) = Ind_{N_1}(M)$, and the restriction to P_1 of the left coaction of P on $Ind_M(N_1)$ coincides with the left coaction of P_1 on $Ind_{N_1}(M)$.*

Proof. The fact that $\text{Ind}_M(N_1) = \text{Ind}_{N_1}(M)$ is immediate from their respective definitions. We denote this common von Neumann algebra again with N.

Now from the proof of 8.2.4, it follows that the inclusions $\widehat{M_1} \subseteq \widehat{M}$ and $\hat{P}_1 \subseteq \hat{P}$ in fact come from an inclusion of linking von Neumann algebraic quantum groupoids $\hat{Q}_1 \subseteq \hat{Q}$. Completely similar as to the situation for von Neumann algebraic quantum groups, this implies that we have a canonical left (translation) coaction of Q_1 on Q, splitting into separate morphisms $\gamma^k_{Q,ij} : Q_{ij} \to (Q_1)_{ik} \otimes Q_{kj}$, and a canonical right (translation) coaction of Q_1 on Q, splitting into separate morphisms $\alpha^k_{Q,ij} : Q_{ij} \to Q_{ik} \otimes (Q_1)_{kj}$, both equivariant with respect to Δ_Q.

Then $\gamma^2_{Q,22} = \gamma_M$, by definition. Now a completely standard argument shows that

$$\gamma^2_{12}(\pi_N(N)) = \{z \in N_1 \otimes M \mid ((\Delta_{N_1})^2_{12} \otimes \iota_M)(z) = (\iota_{N_1} \otimes \gamma^2_{Q,22})(z)\}.$$

Hence, again by definition, $N = \gamma^2_{12}(\pi_N(N))$. Then by equivariance, the left coaction of P_1 on $N = \mathrm{Ind}_{N_1}(M)$ corresponds under π_N to the left coaction γ^1_{12} of P_1 on π_N. Since also $\gamma^1_{Q,11} = \gamma_P$, the canonical left coaction of P_1 on P, equivariance lets us conclude that

$$(\iota_{P_1} \otimes \gamma_N)(\gamma_{N_1} \otimes \iota_M) = (\gamma_P \otimes \iota_N)\gamma_N$$

on $N \subseteq N_1 \otimes M$, which, by the definition of the restriction of a coaction, concludes the proof.

□

For the following definition, recall that a *short exact sequence of von Neumann algebraic quantum groups* (cf. Definition 3.2 of [88]) $M_1 \to M \to M_2$ consists of three von Neumann algebraic quantum groups M_1, M and M_2, such that M_1 is a closed quantum subgroup of M, $\widehat{M_2}$ is a closed quantum subgroup of $\widehat{M}$, and, denoting by γ_M the canonical left coaction of M_2 on M, we have $M_1 = M^{\gamma_M}$. (We note that then also $M_1 = M^{\alpha_M}$, where α_M is the canonical right coaction of M_2 on M, using Proposition 3.1 of [88] and the fact that M_1 is invariant under R_M by Proposition 10.5 of [6].)

Proposition 8.2.6. *Let $M_1 \to M \to M_2$ be a short exact sequence of von Neumann algebraic quantum groups. Suppose that (N_2, α_{N_2}) is a right Galois object for M_2. Denote*

$$(N, \alpha_N) = (Ind_{\widehat{M}}(N_2), Ind_{\widehat{M}}(\alpha_{N_2}))$$

and

$$(N_1, \alpha_{N_1}) = (Red_{M_1}(N), Red_{M_1}(\alpha_N)).$$

Then we have a canonical isomorphism of right M_1-Galois objects between (N_1, α_{N_1}) and (M_1, Δ_{M_1}). Moreover, by reflecting, we then obtain a short exact sequence

$$(P_1 =)M_1 \to P \to P_2$$

of von Neumann algebraic quantum groups.

Proof. Denote by γ_M the canonical left coaction of M_2 on M. Then N_1 consists of those $z \in N_2 \otimes M$ such that $(\alpha_{N_2} \otimes \iota_M)(z) = (\iota_{N_2} \otimes \gamma_M)(z)$ *and* $(\iota_N \otimes \Delta_M)(z) \in N \otimes M \otimes M_1$. Now if $x \in M$ and $\Delta_M(x) \in M \otimes M_1$, then $x \in M_1$. Hence if $z \in N_1$, then $z \in N_2 \otimes M_1$ by the second condition on such elements. But also $M_1 = M^{\gamma_M}$. Hence by the *first* condition on an element $z \in N_1$, we deduce that $z \in N_2^{\alpha_{N_2}} \otimes M_1 = \mathbb{C} \otimes M_1$. This provides a canonical isomorphism $N_1 \to M_1$. It is easily seen that $(N_1, \alpha_1) \cong (M_1, \Delta_{M_1})$ as right Galois objects under this isomorphism. Hence we also obtain a canonical isomorphism between the von Neumann algebraic quantum groups P_1 and M_1.

Now by the Propositions 8.1.3 and 8.2.4, we have that $\widehat{P}_2$ is a closed quantum subgroup of $\widehat{P}$, and P_1 a closed quantum subgroup of P. To end the proof, we should show that if γ_P denotes the canonical left coaction of P_2 on P, then $P_1 = P^{\gamma_P}$. This is equivalent with proving that $P \cap \widehat{P}_2' = P_1$ on $\mathscr{L}^2(P)$.

Now we can place $B(\mathscr{L}^2(P))$ inside $B(\mathscr{L}^2(N) \otimes \mathscr{L}^2(O))$, sending $x \in P$ to $\beta_P(x)$ and $x \in \widehat{P}$ to $x \otimes 1$. Then we should show that $\beta_P(P) \cap (\widehat{P}_2' \otimes 1) = \beta_P(P_1)$. For this, it is sufficient to prove that $N \cap \widehat{P}_2' = N_1$ (which equals $1 \otimes M_1$). Indeed, if then $z \in P$ and the first leg of $\beta_P(z) \in N \otimes O$ commutes with $\widehat{P}_2$, then $\beta_P(z) \in N_1 \otimes O$. Since $(\alpha_N \otimes \iota_O) = (\iota_N \otimes \gamma_O)$ on the range of β_P, we then also have $\beta_P(z) \in N_1 \otimes O_1$. But then $z \in P_1$ by the specific way the imbedding $P_1 \to P$ was defined in Proposition 8.1.3.

So we are left to proving that $N \cap \widehat{P}_2' = N_1$. Now by Lemma 8.1.1 and Lemma 8.2.5,

$$N \cap \widehat{P}_2' = \{x \in N \subseteq N_2 \otimes M \mid (\gamma_{N_2} \otimes \iota_M)(x) = 1_{P_2} \otimes x\}.$$

Since γ_{N_2} is ergodic, we must have $x = 1_{N_2} \otimes m$ for some $m \in M$ when $x \in N \cap \widehat{P}_2'$. But $(1 \otimes M) \cap N = 1 \otimes M_1$, which concludes the proof.

□

Chapter 9

Application: Twisting by 2-cocycles

In the first section of this chapter, we will study a specific class of Galois objects, namely those obtained by twisting with a 2-cocycle. On the dual side, this corresponds to those linking von Neumann algebraic quantum groupoids built upon an identity linking von Neumann algebra. In the second section, we show the relation between Galois objects for the tensor product of two von Neumann algebraic quantum groups and the Galois objects of its constituents.

9.1 2-cocycles

Let M be a von Neumann algebraic quantum group, and let $\Omega \in \widehat{M} \otimes \widehat{M}$ be a unitary 2-cocycle, i.e. a unitary element satisfying

$$(1 \otimes \Omega)(\iota \otimes \Delta_{\widehat{M}})(\Omega) = (\Omega \otimes 1)(\Delta_{\widehat{M}} \otimes \iota)(\Omega).$$

Denote by $\check{\alpha}$ the trivial left coaction $\mathbb{C} \to \widehat{M} \otimes \mathbb{C}$ of $\widehat{M}$. Then $(\check{\alpha}, \Omega)$ is a cocycle action ([88], Definition 1.1). Let

$$N = \widehat{M} \underset{\Omega}{\ltimes} \mathbb{C} := [(\omega \otimes \iota)(W_{\widehat{M}}\Omega^*) \mid \omega \in \widehat{M}_*]^{\sigma-\text{weak}}$$

be the cocycle crossed product ([88], Definition 1.3). (Actually, one should take the von Neumann algebra generated by elements of this last set, in stead of just the σ-weak closure, but it will follow from our Lemma 7.2.6 and the following proposition that this is the same.) Then there is a canonical *right* ergodic coaction α_Ω of M on N, determined by

$$\alpha_\Omega((\omega\otimes\iota)(W_{\widehat{M}}\Omega^*)) = (\omega\otimes\iota\otimes\iota)((W_{\widehat{M}})_{13}(W_{\widehat{M}})_{12}\Omega^*_{12}),$$

where $\omega \in \widehat{M}_*$ ([88], Proposition 1.4 and Theorem 1.11.1). Furthermore, it is integrable ([88], the remark following Lemma 1.12), and we can take the GNS construction for φ_N in $\mathscr{L}^2(M)$, by defining

$$\Lambda_{\varphi_N}((\omega\otimes\iota)(W_{\widehat{M}}\Omega^*)) := \Lambda_M((\omega\otimes\iota)(W_{\widehat{M}}))$$

for $\omega \in \widehat{M}_*$ well-behaved ([88], Proposition 1.15). Finally, (N,α_Ω) is a right Galois object for M, since the unitary $W_{\widehat{M}}\Omega^* \in B(\mathscr{L}^2(M))\otimes N$ satisfies

$$(\iota\otimes\alpha_\Omega)(W_{\widehat{M}}\Omega^*) = (W_{\widehat{M}})_{13}(W_{\widehat{M}})_{12}\Omega^*_{12},$$

so that α_Ω is semi-dual (see Example 7.1.2).

Definition 9.1.1. *A right M-Galois object N is called a* cleft Galois object *for M if there exists a unitary 2-cocycle $\Omega \in \widehat{M}\otimes\widehat{M}$ such that $N \cong (\widehat{M} \ltimes_\Omega \mathbb{C}, \alpha_\Omega)$.*

The following proposition is not very surprising.

Proposition 9.1.2. *Let $\Omega \in \widehat{M}\otimes\widehat{M}$ be a unitary 2-cocycle for a von Neumann algebraic quantum group $\widehat{M}$, and let N be the associated right M-Galois object. Then the Galois map $\tilde{G}$ equals $W_{\widehat{M}}\Omega^*$.*

Proof. Choose $\xi,\eta,\zeta \in \mathscr{L}^2(M)$, and an orthonormal basis ξ_i of $\mathscr{L}^2(M)$. Further, let $m \in \widehat{M}$ be in the Tomita algebra for $\varphi_{\widehat{M}}$, and denote $\omega' = \omega_{\zeta,\Lambda_{\widehat{M}}(m)}$. Then by Proposition 1.15 of [88], $(\omega'\otimes\iota)(W_{\widehat{M}}\Omega^*) \in \mathscr{N}_{\varphi_N}$, $(\omega'\otimes\iota)(W_{\widehat{M}}) \in \mathscr{N}_{\varphi_M}$ and

$$\Lambda_N((\omega'\otimes\iota)(W_{\widehat{M}}\Omega^*)) = \Lambda_M((\omega'\otimes\iota)(W_{\widehat{M}})).$$

So

$$\begin{aligned}
&(\iota\otimes\omega_{\xi,\eta})(\tilde{G})\ \Lambda_M((\omega'\otimes\iota)(W_{\widehat{M}}))\\
&\qquad = (\iota\otimes\omega_{\xi,\eta})(\tilde{G})\Lambda_N((\omega'\otimes\iota)(W_{\widehat{M}}\Omega^*))\\
&\qquad = \Lambda_M((\omega_{\xi,\eta}\otimes\iota)(\alpha_\Omega((\omega'\otimes\iota)(W_{\widehat{M}}\Omega^*))))\\
&\qquad = \Lambda_M((\omega'\otimes\omega_{\xi,\eta}\otimes\iota)((W_{\widehat{M}})_{13}(W_{\widehat{M}})_{12}\Omega^*_{12}))\\
&\qquad = \Lambda_M(\sum_i(\omega'\otimes\omega_{\xi,\xi_i}\otimes\omega_{\xi_i,\eta}\otimes\iota)((W_{\widehat{M}})_{14}(W_{\widehat{M}})_{13}\Omega^*_{12})),
\end{aligned}$$

where the sum is taken in the σ-strong-topology.

On the other hand, using Result 8.6 of [56], adapted to the von Neumann algebra setting, we get

$$\begin{aligned}(\iota\otimes\omega_{\xi,\eta})(W_{\widehat{M}}\Omega^*)\ \Lambda_M((\omega'\otimes\iota)(W_{\widehat{M}}))\\ &= \sum_i(\iota\otimes\omega_{\xi_i,\eta})(W_{\widehat{M}})(\iota\otimes\omega_{\xi,\xi_i})(\Omega^*)\Lambda_M((\omega'\otimes\iota)(W_{\widehat{M}}))\\ &= \sum_i\Lambda_M((\omega_{\xi_i,\eta}\otimes\iota)\Delta_M((\omega'(\cdot(\iota\otimes\omega_{\xi,\xi_i})(\Omega^*))\otimes\iota)(W_{\widehat{M}})))\\ &= \sum_i\Lambda_M((\omega'\otimes\omega_{\xi,\xi_i}\otimes\omega_{\xi_i,\eta}\otimes\iota)((W_{\widehat{M}})_{14}(W_{\widehat{M}})_{13}\Omega^*_{12})),\end{aligned}$$

so that the result follows by the closedness of Λ_M and the density of elements of the form $\Lambda_M((\omega'\otimes\iota)(W_{\widehat{M}}))$ in $\mathscr{L}^2(M)$.

□

Proposition 9.1.3. *Under the map LQG from Galois objects to linking von Neumann algebraic quantum groupoids, cleft Galois objects correspond precisely to those linking von Neumann algebraic quantum groupoids whose underlying linking von Neumann algebra is the identity.*

Proof. Let Ω be a unitary 2-cocycle for a von Neumann algebraic quantum group $\widehat{M}$, and N the associated right M-Galois object. We already know that $\mathscr{L}^2(N)$ can be identified with $\mathscr{L}^2(M)$. It is also not difficult to see that under this correspondence, the unitary implementation U of α_N becomes just the regular right multiplicative unitary V_M: Take again $m\in\widehat{M}$ in the Tomita algebra for $\varphi_{\widehat{M}}$, take $\zeta\in\mathscr{L}^2(M)$, and denote $\omega'=\omega_{\zeta,\Lambda_{\widehat{M}}(m)}$. Then for $\omega\in M_*$ such that $\omega(\,\cdot\,\delta_M^{-1})$ extends to a bounded normal functional ω_δ on M, we have

$$\begin{aligned}&(\iota\otimes\omega)(U)\Lambda_N((\omega'\otimes\iota)(W_{\widehat{M}}\Omega^*))\\ &\quad= \Lambda_N((\omega'(\,\cdot\,(\iota\otimes\omega_\delta)(W_{\widehat{M}}))\otimes\iota)(W_{\widehat{M}}\Omega^*))\\ &\quad= \Lambda_M((\omega'(\,\cdot\,(\iota\otimes\omega_\delta)(W_{\widehat{M}}))\otimes\iota)(W_{\widehat{M}}))\\ &\quad= (\iota\otimes\omega)(V_M)\Lambda_M((\omega'\otimes\iota)(W_{\widehat{M}}))\\ &\quad= (\iota\otimes\omega)(V_M)\Lambda_N((\omega'\otimes\iota)(W_{\widehat{M}}\Omega^*)).\end{aligned}$$

Hence $\mathscr{L}^2(N)$, as a right $\widehat{M}$-module, is just $\mathscr{L}^2(M)$ with its natural right $\widehat{M}$-module structure. Hence $(\widehat{Q},e)$ will be the identity linking von Neumann algebra for $\widehat{M}$.

Conversely, suppose that $(\widehat{Q}, e)$ is a linking von Neumann algebraic quantum groupoid built upon the identity linking von Neumann algebra for the underlying von Neumann algebra of some von Neumann algebraic quantum group $\widehat{M}$. Then in particular, $\widehat{Q}_{12} = \widehat{M}$. Put $\Omega = \hat{\Delta}_{12}(1_{\widehat{M}})$. Then for $x \in \widehat{M}$, we have $\hat{\Delta}_{12}(x) = \Omega \cdot \Delta_{\widehat{M}}(x)$. So from the coassociativity of $\hat{\Delta}_{12}$, it follows immediately that Ω satisfies the 2-cocycle relation. Moreover, it is unitary, since $\hat{\Delta}_{12}(1_{\widehat{M}})^* = \hat{\Delta}_{21}(1_{\widehat{M}})$ and $\hat{\Delta}_{12}(1_{\widehat{M}})\hat{\Delta}_{21}(1_{\widehat{M}}) = \hat{\Delta}_{11}(1_{\widehat{M}})$, and $\hat{\Delta}_{21}(1_{\widehat{M}})\hat{\Delta}_{12}(1_{\widehat{M}}) = \hat{\Delta}_{22}(1_{\widehat{M}})$.

Further, since $\hat{\Lambda}_{12} = \hat{\Lambda}_{22}$ in this case, we have for $x, y \in \mathscr{N}_{\varphi_M}$ that

$$\begin{aligned}(\widehat{W}^2_{12})^*(\Lambda_{\widehat{M}}(x) \otimes \Lambda_{\widehat{M}}(y)) &= (\Lambda_{\widehat{M}} \otimes \Lambda_{\widehat{M}})(\hat{\Delta}_{12}(y)(x \otimes 1)) \\ &= \Omega \cdot (\Lambda_{\widehat{M}} \otimes \Lambda_{\widehat{M}})(\Delta_{\widehat{M}}(y)(x \otimes 1)) \\ &= \Omega \cdot W^*_{\widehat{M}} \cdot (\Lambda_{\widehat{M}}(x) \otimes \Lambda_{\widehat{M}}(y)),\end{aligned}$$

from which it follows that for the right M-Galois object N associated to $(\widehat{Q}, e)$, the Galois unitary $\tilde{G}$ equals $W_{\widehat{M}}\Omega^*$. Since N is equal to the σ-weak closure of the first leg of $\tilde{G}$, and since $(\iota \otimes \alpha_N)(\tilde{G}) = (W_{\widehat{M}})_{13}\tilde{G}_{12}$, it follows that N is just the cleft Galois object associated to Ω.

□

Note that when reconstructing a linking von Neumann algebraic quantum groupoid from a right cleft M-Galois object N, we will always identify $\widehat{M}$ with $\widehat{P}$ by first identifying $\mathscr{L}^2(N)$ with $\mathscr{L}^2(M)$ in the manner recalled at the beginning, and then taking the standard left representation of $\widehat{M}$ on $\mathscr{L}^2(M)$.

We will now also call such linking von Neumann algebraic quantum groupoids with underlying identity linking algebra *cleft linking von Neumann algebraic quantum groupoids*, and the associated bi-Galois objects *cleft bi-Galois objects.*

Corollary 9.1.4. *Let $\widehat{M}$ be a von Neumann algebraic quantum group, and Ω a unitary 2-cocycle in $\widehat{M} \otimes \widehat{M}$. Then the Ω-twisted Hopf-von Neumann algebra $(\widehat{M}, \hat{\Delta}_\Omega)$, where*

$$\hat{\Delta}_\Omega(m) = \Omega\Delta_{\widehat{M}}(m)\Omega^*,$$

is a von Neumann algebraic quantum group.

Proof. This follows straightforwardly from the proof of the previous proposition. For then we have that in the linking von Neumann algebraic quantum groupoid $(\hat{Q}, e)$ for the cleft right Galois object N associated to Ω, the corner $\hat{P} = \hat{Q}_{11}$ equals $\widehat{M}$, equipped with the coproduct

$$\begin{aligned} \hat{\Delta}_{11}(x) &= \hat{\Delta}_{12}(1)\hat{\Delta}_{22}(x)\hat{\Delta}_{21}(1) \\ &= \Omega\Delta_{\widehat{M}}\Omega^{*}. \end{aligned}$$

So by Theorem 7.3.7, $(\widehat{M}, \hat{\Delta}_{\Omega})$ is a von Neumann algebraic quantum group. □

Remark: Corollary 9.1.4 answers negatively a question of [46]: the 2-pseudo-cocycles Ω_q of [46] are *not* 2-cocycles, since $SU_0(2)$ is not a quantum group. This of course does not rule out the possibility that the $SU_q(2)$ *are* cocycle twists of each other in some other way.

Proposition 9.1.3 also shows that there is no ambiguity in the definition of a cleft bi-Galois object: if (N, γ_N, α_N) is a bi-Galois object, and the associated right Galois object N is cleft with 2-cocycle Ω, then (N, γ_N) will be cleft with 2-cocycle Ω^*.

Finally, remark that the reduced C*-algebra $\hat{D}$ of the reflection $\hat{P}$ of $\widehat{M}$ across a cleft M-Galois object will in general *not* be the same as the reduced C*-algebra $\hat{A}$ of $\widehat{M}$, as the example in section 10.3 will show. However, it will still be C*-Morita equivalent to the original one, by the results of the final section of the sixth chapter.

For the following proposition, recall that if $\widehat{M}$ is a von Neumann algebraic quantum group and Ω_1, Ω_2 are two unitary 2-cocycles in $\widehat{M} \otimes \widehat{M}$, then Ω_1 and Ω_2 are called *cohomologous* if there exists a unitary $v \in \widehat{M}$ such that $(v \otimes v)\Omega_1 = \Omega_2 \cdot \Delta_{\widehat{M}}(v)$. We will call Ω_1 and Ω_2 *centrally* cohomologous when we can choose $v \in \mathscr{Z}(\widehat{M})$.

Proposition 9.1.5. *Let M be a von Neumann algebraic quantum group. Then*

1. *two cleft right Galois objects are isomorphic iff the associated 2-cocycles are cohomologous, and*

2. *two cleft bi-Galois objects are isomorphic iff the associated 2-cocycles are centrally cohomologous.*

Proof. Suppose two isomorphic cleft right Galois objects N_1 and N_2 are given, with respective associated 2-cocycles Ω_1 and Ω_2. Let Φ be the associated isomorphism between the respective linking von Neumann algebraic quantum groupoids $\widehat{Q}_1$ and $\widehat{Q}_2$. Put $u = \Phi(e_{12}) \in \widehat{M}$. Then u will be a unitary, and

$$\Phi(\begin{pmatrix} x & y \\ z & w \end{pmatrix}) = \begin{pmatrix} uxu^* & uy \\ zu^* & w \end{pmatrix}. \tag{9.1}$$

Since $(\Phi \otimes \Phi)\Delta_{\hat{N}_1}(x) = \Delta_{\hat{N}_2}(\Phi(x))$ for $x \in \widehat{M}$, we get

$$\begin{aligned} (u \otimes u)\Omega_1 &= (u \otimes u)\Delta_{\hat{N}_1}(1_{\widehat{M}}) \\ &= (\Phi \otimes \Phi)\Delta_{\hat{N}_1}(1_{\widehat{M}}) \\ &= \Delta_{\hat{N}_2}(\Phi(1_{\widehat{M}})) \\ &= \Omega_2\Delta_{\widehat{M}}(u), \end{aligned}$$

and hence Ω_1 and Ω_2 are cohomologous. If N_1 and N_2 are isomorphic cleft bi-Galois objects, then we must also have $uxu^* = x$ for all $x \in \widehat{M}$, hence Ω_1 and Ω_2 centrally cohomologous.

Conversely, given two 2-cocycles which are (centrally) cohomologous by some (central) unitary $u \in \widehat{M}$, it is clear that if we define Φ by the formula 9.1, this will be an isomorphism between the corresponding linking von Neumann algebraic quantum groupoids, whose dual will be an isomorphism of the corresponding (bi-)Galois objects.

□

This proposition shows that the set of equivalence classes of 2-cocycles, under the equivalence relation of being centrally cohomologous, can be imbedded in the groupoid constructed in section 7.5. It is easy to see that the composition of two cleft bi-Galois objects is again cleft, with the product of the two associated 2-cocycles (in the proper order) as the associated 2-cocycle. Hence the set of isomorphism classes of cleft bi-Galois objects forms a subgroupoid of the '2-cohomology groupoid' of section 7.5.

Proposition 9.1.6. *Let M be a von Neumann algebraic quantum group, $\Omega \in \widehat{M} \otimes \widehat{M}$ a unitary 2-cocycle, and (N, α_N) the associated cleft right M-Galois object.*

1. *The one-parametergroups $\tau_t^{\widehat{M}}$ and $\tau_t^{\hat{P}}$ on $\widehat{M}$ are cocycle equivalent.*

2. *The 2-cocycles* Ω *and* $(\tau_t^{\widehat{M}} \otimes \tau_t^{\widehat{M}})(\Omega)$ *are cohomologous.*
3. *The 2-cocycles* Ω *and* $\tilde{\Omega} := (R_{\widehat{M}} \otimes R_{\widehat{M}})(\Sigma\Omega^*\Sigma)$ *are cohomologous.*

Proof. Denote by $u_t = \nabla_{\hat{N}}^{it}\nabla_{\widehat{M}}^{-it} \in \widehat{M}$ the cocycle derivative of $\varphi_{\hat{P}}$ with respect to $\varphi_{\widehat{M}}$, so that $u_{s+t} = u_s\sigma_s^{\widehat{M}}(u_t)$. Denote $v_t = \nabla_N^{it}\nabla_M^{-it}$. Then also $v_t \in \widehat{M}$, since ∇_N^{it} and ∇_M^{it} implement the same automorphism on $\widehat{M}'$. Finally, denote $X = J_N J_M$, then X is a unitary in $\widehat{M}$ for the same reason.

We show that the one-parametergroup v_t is a 1-cocycle with respect to $\tau_t^{\widehat{M}}$.

By Lemma 7.2.15 and Proposition 9.1.2, we have

$$(\nabla_M^{it} \otimes u_t\nabla_{\widehat{M}}^{it})(W_{\widehat{M}}\Omega^*) = (W_{\widehat{M}}\Omega^*)(\nabla_N^{it} \otimes u_t\nabla_{\widehat{M}}^{it}).$$

Since $\nabla_M^{it} \otimes \nabla_{\widehat{M}}^{it}$ commutes with $W_{\widehat{M}}$ and ∇_M^{it} implements $\tau_t^{\widehat{M}}$ on $\widehat{M}$, the left hand side can be rewritten as $(1 \otimes u_t)W_{\widehat{M}}(\tau_t^{\widehat{M}} \otimes \sigma_t^{\widehat{M}})(\Omega^*)(\nabla_M^{it} \otimes \nabla_{\widehat{M}}^{it})$, and so, bringing $W_{\widehat{M}}$ and $(\nabla_M^{it} \otimes \nabla_{\widehat{M}}^{it})$ to the other side, we obtain

$$\Delta_{\widehat{M}}(u_t)(\tau_t^{\widehat{M}} \otimes \sigma_t^{\widehat{M}})(\Omega^*) = \Omega^*(v_t \otimes u_t).$$

Hence

$$\begin{aligned} v_{s+t} \otimes u_{s+t} &= \Omega\Delta_{\widehat{M}}(u_{s+t})(\tau_{s+t}^{\widehat{M}} \otimes \sigma_{s+t}^{\widehat{M}})(\Omega^*) \\ &= \Omega\Delta_{\widehat{M}}(u_s\sigma_s^{\widehat{M}}(u_t))(\tau_{s+t}^{\widehat{M}} \otimes \sigma_{s+t}^{\widehat{M}})(\Omega^*) \\ &= \Omega\Delta_{\widehat{M}}(u_s)(\tau_s^{\widehat{M}} \otimes \sigma_s^{\widehat{M}})(\Omega^*) \\ &\quad \cdot(\tau_s^{\widehat{M}} \otimes \sigma_s^{\widehat{M}})(\Omega\Delta_{\widehat{M}}(u_t)(\tau_t^{\widehat{M}} \otimes \sigma_t^{\widehat{M}})(\Omega^*)) \\ &= v_s\tau_s^{\widehat{M}}(v_t) \otimes u_s\sigma_s^{\widehat{M}}(u_t), \end{aligned}$$

from which the cocycle property of v_t follows.

Then $\tau_t^{\hat{P}}$ will be cocycle equivalent with $\tau_t^{\widehat{M}}$ by v_t, since $\tau_t^{\hat{N}}$ is implemented by ∇_N^{it}.

Now note that v_t also equals $P_N^{it}P_M^{-it}$ (by definition of P_N). So using the third equality of Corollary 7.2.7,

$$W_{\widehat{M}}\Omega^*(v_t \otimes v_t)(P_M^{it} \otimes P_M^{it}) = (P_M^{it} \otimes v_tP_M^{it})W_{\widehat{M}}\Omega^*.$$

Using that $P_M^{it} = P_{\widehat{M}}^{it}$, taking $W_{\widehat{M}}$ and $P_M^{it}\otimes P_M^{it}$ to the other side, and using that $P_M^{it}\otimes P_M^{it}$ commutes with $W_{\widehat{M}}$, we arrive at

$$\Omega^*(v_t\otimes v_t) = \Delta_{\widehat{M}}(v_t)(\tau_t^{\widehat{M}}\otimes\tau_t^{\widehat{M}})(\Omega^*),$$

which proves the second statement.

Finally, as observed already in Proposition 9.1.3, the unitary implementation of α_N is just V_M itself. So by Lemma 7.2.4, we have

$$W_{\widehat{M}}\Omega^*(J_N\otimes J_N)\Sigma = \Sigma V_M\Sigma(J_{\widehat{M}}\otimes J_N)W_{\widehat{M}}\Omega^*.$$

Multiplying to the right with $(J_M\otimes J_M)\Sigma$, we get

$$\begin{aligned}
W_{\widehat{M}}\Omega^*(X\otimes X) &= \Sigma V_M\Sigma(1\otimes X)(J_{\widehat{M}}\otimes J_M)W_{\widehat{M}}\Omega^*(J_M\otimes J_M)\Sigma\\
&= \Sigma V_M\Sigma(1\otimes X)(J_{\widehat{M}}\otimes J_M)W_{\widehat{M}}(J_M\otimes J_M)\Sigma\tilde{\Omega}^*\\
&= \Sigma V_M\Sigma(1\otimes X)(J_{\widehat{M}}\otimes J_M)\Sigma V_M\Sigma(J_{\widehat{M}}\otimes J_M)W_{\widehat{M}}\tilde{\Omega}^*\\
&= \Sigma V_M\Sigma(1\otimes X)\Sigma V_M^*\Sigma W_{\widehat{M}}\tilde{\Omega}^*\\
&= (1\otimes X)W_{\widehat{M}}\tilde{\Omega}^*,
\end{aligned}$$

from which $\Omega^*(X\otimes X) = \Delta_{\widehat{M}}(X)\tilde{\Omega}^*$ immediately follows. □

We have the following formula for the multiplicative unitary $\widehat{W}_\Omega$ of $(\widehat{M},\hat{\Delta}_\Omega)$.

Proposition 9.1.7. *Let $\widehat{M}$ be a von Neumann algebraic quantum group, and Ω a unitary 2-cocycle in $\widehat{M}\otimes\widehat{M}$. Then the left regular multiplicative unitary $W_{\hat{P}}$ of the reflected von Neumann algebraic quantum group $\hat{P}$ equals*

$$W_{\hat{P}} = (J_N\otimes J_{\widehat{M}})\Omega W_{\widehat{M}}^*(J_M\otimes J_{\widehat{M}})\Omega^*.$$

Proof. Since the underlying linking von Neumann algebra of the associated linking von Neumann algebraic quantum groupoid is the identity linking von Neumann algebra $\widehat{M}\otimes M_2(\mathbb{C})$, we can identify the GNS-construction with $\begin{pmatrix}\mathscr{L}^2(\widehat{M}) & \mathscr{L}^2(\widehat{M})\\ \hline \mathscr{L}^2(\widehat{M}) & \mathscr{L}^2(\widehat{M})\end{pmatrix}$, and we then have that $\hat{\Lambda}_{12}$ equals $\Lambda_{\widehat{M}}$, while $\hat{\Lambda}_{21}$ becomes $\Lambda_{\hat{P}}$. Then from Lemma 7.3.6, and the fact that $\hat{\Delta}_{21}(x) = \Delta_{\widehat{M}}(x)\Omega^*$, we conclude that

$$\begin{aligned}
W_{\hat{P}}\Omega &= ((J_M\otimes J_{\widehat{M}})\tilde{G}(J_N\otimes J_{\widehat{M}}))^*\\
&= (J_N\otimes J_{\widehat{M}})(\Omega W_{\widehat{M}}^*)(J_M\otimes J_{\widehat{M}}).
\end{aligned}$$

The proposition follows.

□

The following is related to Proposition 4.5 of [10].

Proposition 9.1.8. *Let M, P_1 and P_2 be von Neumann algebraic quantum groups. Let $(N_1, \gamma_1, \alpha_1)$ be a P_1-M-bi-Galois object, and $(N_2, \gamma_2, \alpha_2)$ a P_2-M-bi-Galois object. Suppose that $\mathscr{L}^2(N_1)$ and $\mathscr{L}^2(N_2)$ are isomorphic as right $\widehat{M}$-modules. Then there exists a 2-cocycle Ω of $\widehat{P}_2$ such that, with $(N_\Omega, \gamma_\Omega, \alpha_\Omega)$ the natural cleft bi-Galois object associated to Ω, the bi-Galois object $(N_1, \gamma_1, \alpha_1)$ is isomorphic to the composition of $(N_\Omega, \gamma_\Omega, \alpha_\Omega)$ and $(N_2, \gamma_2, \alpha_2)$.*

Proof. By the theory in section 5.5, it is easy to see that the commutant of the direct sum right $\widehat{M}$-representations on $\mathscr{L}^2(N_1)$ and $\mathscr{L}^2(N_2)$ will be isomorphic to the linking von Neumann algebra underlying the composition of $(N_1, \gamma_1, \alpha_1)$ and the inverse of $(N_2, \gamma_2, \alpha_2)$. Since the right $\widehat{M}$-representations are isomorphic, this composite linking von Neumann algebra will be isomorphic to the identity linking von Neumann algebra. Hence its associated bi-Galois structure is cleft, and the proposition follows.

□

Corollary 9.1.9. *If M is a von Neumann algebraic quantum group with $\widehat{M}$ a properly infinite factor with separable predual, then any right M-Galois object (whose underlying von Neumann algebra is separable) is cleft.*

Proof. By the previous proposition, this is clear if $\widehat{M}$ is type III, since there is then only one separable right $\widehat{M}$-module up to isomorphism. Also, since a right Galois object N for a finite-dimensional von Neumann algebraic quantum group is finite-dimensional itself (see the remark on page 224), the proposition is also clear for the type I_∞ case.

We are left with the type II_∞ case. For this it is enough to prove, that if N is a right Galois object for M with $\widehat{M}$ type II_1, then also the von Neumann algebra $\widehat{P}$ of the reflected von Neumann algebraic quantum group is type II_1. Now by Theorem 9 of [37], we know that $\widehat{M}$ will be a compact quantum group, with its unique tracial state as the (left and right) invariant state. Hence $\widehat{P}$ is also a compact quantum group of Kac type by Proposition 10.3.2, and so $\widehat{P}$ is type II_1.

□

9.2 Generalized quantum doubles

We now treat a very special type of 2-cocycle. Let M_1 and M_2 be two von Neumann algebraic quantum groups, and let $Z \in \widehat{M}_1 \otimes \widehat{M}_2$ be a bicharacter in the sense that

$$(\hat{\Delta}_1 \otimes \iota)(Z) = Z_{13}Z_{23},$$

$$(\iota \otimes \hat{\Delta}_2)(Z) = Z_{13}Z_{12}.$$

Then it is easily checked that

$$\Omega_Z := (\Sigma Z \Sigma)_{23} \in (\widehat{M}_1 \otimes \widehat{M}_2) \otimes (\widehat{M}_1 \otimes \widehat{M}_2)$$

is a unitary 2-cocycle for the tensor product von Neumann algebraic quantum group $M = M_1 \otimes M_2$ (whose comultiplication is $\mathrm{Ad}(\Sigma)_{23}(\Delta_{\widehat{M}_1} \otimes \Delta_{\widehat{M}_2})$).

Definition 9.2.1. *(cf. section 8 of [4]) In the above situation, we call the Ω_Z-twisted von Neumann algebraic quantum group of $M = M_1 \otimes M_2$ the* generalized quantum double *(of M_1 and M_2 with respect to Z), and denote it as M_Z.*

We also denote $\widehat{M}_Z$ then for $\widehat{M_Z}$.

The following result was proven for Hopf algebras in Proposition 12 of [71].

Proposition 9.2.2. *Let M_1, M_2 and P be von Neumann algebraic quantum groups, and put $M = M_1 \otimes M_2$. Let N be a P-M-bi-Galois object. Then there exist two von Neumann algebraic quantum groups P_1 and P_2, a bicharacter $Z \in \hat{P}_1 \otimes \hat{P}_2$, and P_i-M_i-bi-Galois objects $(N_i, \alpha_i, \gamma_i)$, such that N is isomorphic to the composition of the $P_1 \otimes P_2$-$M_1 \otimes M_2$-bi-Galois object $N_1 \otimes N_2$ with the canonical P_Z-$(P_1 \otimes P_2)$-bi-Galois object.*

Proof. Denote

$$N_1 = \{x \in N \mid \alpha_N(x) \in N \otimes (M_1 \otimes 1)\},$$

$$N_2 = \{x \in N \mid \alpha_N(x) \in N \otimes (1 \otimes M_2)\}.$$

Define $\alpha_1 : N_1 \to N_1 \otimes M_1$ by $\alpha_1(x) \otimes 1_{M_2} = \alpha_N(x)$, and similarly $\alpha_2 : N_2 \to N_2 \otimes M_2$. Then by Proposition 8.1.3, (N_i, α_i) will be a Galois object for (M_i, Δ_i).

Denote by $\tilde{G}_{N_i}$ the Galois unitary for N_i, and by $\tilde{G}_N$ the Galois unitary for N. Denote by π_i^N the representation of N_i on $\mathscr{L}^2(N)$. Denote $\tilde{G}_n = ((\iota \otimes \pi_1^N)\tilde{G}_{N_1})_{13}((\iota \otimes \pi_2^N)\tilde{G}_{N_2})_{23}$, which is a unitary

$$\mathscr{L}^2(\hat{N}_1) \otimes \mathscr{L}^2(\hat{N}_2) \otimes \mathscr{L}^2(N) \to \mathscr{L}^2(\widehat{M}_1) \otimes \mathscr{L}^2(\widehat{M}_2) \otimes \mathscr{L}^2(N).$$

Then interpreting $(\iota\otimes\alpha_N)\tilde{G}_n$ as an element inside $\hat{O}_1\otimes\hat{O}_2\otimes N\otimes M_1\otimes M_2$, we compute that

$$\begin{aligned}
&(\iota\otimes\alpha_N)\tilde{G}_n\\
&\quad= (W_{\widehat{M}_1})_{14}((\iota\otimes\pi_1^N)\tilde{G}_{N_1})_{13}(W_{\widehat{M}_2})_{25}(\tilde{G}_{N_2})_{23}\\
&\quad= (W_{\widehat{M}_1})_{14}((\iota\otimes\pi_1^N)\tilde{G}_{N_1})_{13}(W_{\widehat{M}_2})_{25}((\iota\otimes\pi_2^N)\tilde{G}_{N_2})_{23}\\
&\quad= (W_{\widehat{M}_1})_{14}(W_{\widehat{M}_2})_{25}((\iota\otimes\pi_1^N)\tilde{G}_{N_1})_{13}((\iota\otimes\pi_2^N)\tilde{G}_{N_2})_{23}\\
&\quad= (W_{\widehat{M}_1})_{1245}(\tilde{G}_n)_{123}.
\end{aligned}$$

We conclude that $(\iota\otimes\alpha_N)(\tilde{G}_n^*\tilde{G}_N) = (\tilde{G}_n^*\tilde{G}_N)\otimes 1_M$, hence $\tilde{G}_N = \tilde{G}_n(u\otimes 1_N)$ for some unitary

$$u:\mathscr{L}^2(N)\to\mathscr{L}^2(\hat{N}_1)\otimes\mathscr{L}^2(\hat{N}_2).$$

Moreover, u is then right $\widehat{M}$-linear. By Proposition 9.1.8, there exists a unitary 2-cocycle $\Omega\in\hat{P}_1\otimes\hat{P}_2$ such that N is isomorphic to the composition of the $P_1\otimes P_2$-$M_1\otimes M_2$-bi-Galois object $N_1\otimes N_2$ with the canonical P_Ω-$(P_1\otimes P_2)$-bi-Galois object.

So to finish the proof, we should show that $\Omega\in(\hat{P}_1\otimes\hat{P}_2)\otimes(\hat{P}_1\otimes\hat{P}_2)$ arises from a bicharacter. For this, we first express u in a more concrete way. Take $x\in\mathscr{N}_{\varphi_{N_1}}$ and $y\in\mathscr{N}_{\varphi_{N_2}}$. Then it is easy to see that $xy\in\mathscr{N}_{\varphi_N}$, for

$$\begin{aligned}
&(\iota\otimes\varphi_M)(\alpha_N(y^*x^*xy))\\
&\quad= (\iota\otimes\varphi_{M_1}\otimes\varphi_{M_2})(\alpha_{N_2}(y)_{13}^*\alpha_{N_1}(x^*x)_{12}\alpha_{N_2}(y))\\
&\quad= (\iota\otimes\varphi_{M_2})(\alpha_{N_2}(y)^*((\iota\otimes\varphi_{M_1})(\alpha_{N_1}(x^*x))\otimes 1)\alpha_{N_2}(y))\\
&\quad= \varphi_{N_1}(x)\varphi_{N_2}(y).
\end{aligned}$$

Since for $x\in\mathscr{N}_{\varphi_N}$, we have

$$\tilde{G}_N(\Lambda_{\varphi_N\otimes\iota_N}(x\otimes 1_N)) = \Lambda_{\varphi_M\otimes\iota_N}(\alpha_N^{\mathrm{op}}(x)),$$

and for $x\in\mathscr{N}_{\varphi_{N_1}}$ and $y\in\mathscr{N}_{\varphi_{N_2}}$, we have

$$\begin{aligned}
&(\tilde{G}_{N_1})_{13}(\tilde{G}_{N_2})_{23}(\Lambda_{\varphi_{N_1}\otimes\varphi_{N_2}\otimes\iota_N}(x\otimes y\otimes 1_N))\\
&\quad= (\Lambda_{\varphi_{M_1}\otimes\varphi_{M_2}\otimes\iota_N})(\alpha_{N_1}^{\mathrm{op}}(x)_{13}\alpha_{N_2}^{\mathrm{op}}(y)_{23}),
\end{aligned}$$

we deduce that

$$u^*(\Lambda_{N_1}(x)\otimes\Lambda_{N_2}(y)) = \Lambda_N(xy)$$

for $x \in \mathscr{N}_{\varphi_{N_1}}$ and $y \in \mathscr{N}_{\varphi_{N_2}}$.

It follows immediately from this that u is left N_1-linear. We also want to show that $uC_N(x) = (1 \otimes C_{N_2}(x))u$ for $x \in N_2$. Clearly, for this it is sufficient to show that $\sigma_t^{\varphi_N} = \sigma_t^{\varphi_{N_2}}$ on N_2. Now let $\alpha_N^{M_2}$ be the restriction of α_N to M_2. Remark that in this case $\alpha_N^{M_2}$ is determined by

$$(\alpha_N \otimes \iota_{M_2})\alpha_N^{M_2}(x) = (\iota_N \otimes \iota_{M_1} \otimes \Delta_{M_2})(\alpha_N(x))$$

for $x \in N$. Hence it is clear that $N^{\alpha_N^{M_2}} = N_1$. Moreover, since for $x \in N^+$, we have

$$(\iota_N \otimes \iota_{M_1} \otimes (\iota_{M_2} \otimes \varphi_{M_2})\Delta_{M_2})(\alpha_N(x)) = (\iota_N \otimes \iota_{M_1} \otimes \varphi_{M_2})\alpha_N(x),$$

we get by Lemma 8.1.1 that

$$T := \alpha_N^{-1} \circ (\iota_N \otimes \iota_{M_1} \circ \varphi_{M_2}) \circ \alpha$$

is an nsf operator valued weight $N \to N_1$. It is also clear then that $\varphi_N = \varphi_{N_1} \circ T$, by Fubini. We conclude by Lemma IX.4.21 of [84] that $\sigma_t^{\varphi_N}$ restricts to $\sigma_t^{\varphi_{N_1}}$ on N_1. By symmetry, $\sigma_t^{\varphi_N}$ restricts to $\sigma_t^{\varphi_{N_2}}$ on N_2, which is what we needed to prove.

Now we can show that the 2-cocycle Ω is in fact of a special form. First note that by the general theory, it will equal the operator

$$(1 \otimes u_{34})\tilde{G}_n^*(1 \otimes u_{34}^*)(\tilde{G}_{N_1})_{13}(\tilde{G}_{N_2})_{2,24}.$$

Since u is left N_1-linear

$$\begin{aligned}
&(1 \otimes u_{34})\tilde{G}_n^*(1 \otimes u_{34}^*)(\tilde{G}_{N_1})_{13}(\tilde{G}_{N_2})_{24}(x \otimes 1 \otimes 1 \otimes 1)\\
&\quad = (1 \otimes u_{34})\tilde{G}_n^*(1 \otimes u_{34}^*)(\alpha_{N_1}^{\mathrm{op}})(x)_{13}(\tilde{G}_{N_1})_{13}(\tilde{G}_{N_2})_{24}\\
&\quad = (1 \otimes u_{34})\tilde{G}_n^*(\alpha_N^{\mathrm{op}})(x)_{13}(1 \otimes u_{34}^*)(\tilde{G}_{N_1})_{13}(\tilde{G}_{N_2})_{24}\\
&\quad = (1 \otimes u_{34})(x \otimes 1 \otimes 1)\tilde{G}_n^*(1 \otimes u_{34}^*)(\tilde{G}_{N_1})_{13}(\tilde{G}_{N_2})_{24}\\
&\quad = (x \otimes 1 \otimes 1 \otimes 1)(1 \otimes u_{34})\tilde{G}_n^*(1 \otimes u_{34}^*)(\tilde{G}_{N_1})_{13}(\tilde{G}_{N_2})_{24}.
\end{aligned}$$

Hence the first leg of Ω lies in $\hat{P}_1 \cap N_1'$. Since this relative commutant is trivial, we deduce that $\Omega \in (1 \otimes \hat{P}_2) \otimes (\hat{P}_1 \otimes \hat{P}_2)$. Now we have also shown that $(1 \otimes C_{N_2}(N_2))u = uC_N(N_2)$. From this, it easily follows that the fourth leg of Ω commutes with $C_{N_2}(N_2)$, so lies in N_2. Since $\hat{P}_2 \cap N_2 = \mathbb{C} \cdot 1_{\hat{P}_2}$, also the fourth leg of Ω is trivial. Hence $\Omega = \Sigma_{23}K_{23}\Sigma_{23}$ for some unitary

$K \in \hat{P}_1 \otimes \hat{P}_2$.

Some calculation with the 2-cocycle identity yields that

$$K_{24}^*(1 \otimes (\iota \otimes \Delta_{\hat{P}_2})(K)) = K_{13}^*((\Delta_{\hat{P}_1} \otimes \iota)(K) \otimes 1),$$

and hence these expressions must equal Z_{23} for some unitary Z. Then

$$(\Delta_{\hat{P}_1} \otimes \iota)(K) = K_{13} Z_{23},$$
$$(\iota \otimes \Delta_{\hat{P}_2})(K) = K_{13} Z_{12}.$$

Using coassociativity, we get

$$\begin{aligned}(\iota \otimes \iota \otimes \Delta_{\hat{P}_2})(\iota \otimes \Delta_{\hat{P}_2})(K) &= (\iota \otimes \iota \otimes \Delta_{\hat{P}_2})(K_{13}) Z_{12} \\ &= K_{14} Z_{13} Z_{12},\end{aligned}$$

while

$$(\iota \otimes \Delta_{\hat{P}_2} \otimes \iota)(\iota \otimes \Delta_{\hat{P}_2})(K) = K_{14}(\iota \otimes \Delta_{\hat{P}_2} \otimes \iota)(Z_{12}),$$

so that $(\iota \otimes \Delta_{\hat{P}_2})(Z) = Z_{13} Z_{12}$. A similar calculation with $\Delta_{\hat{P}_1}$ shows that Z is in fact a bicharacter. But now

$$\begin{aligned}(\iota \otimes \Delta_{\hat{P}_2})(KZ^*) &= K_{13} Z_{12} Z_{12}^* Z_{13}^* \\ &= (KZ)_{13}^*,\end{aligned}$$

and similarly $(\Delta_{\hat{P}_1} \otimes \iota)(KZ^*) = (KZ^*)_{13}$. So

$$\mathrm{Ad}(\Sigma)_{23}((\Delta_{\hat{P}_1} \otimes \Delta_{\hat{P}_2}^{\mathrm{op}})(KZ^*)) = (KZ^*) \otimes 1,$$

which means that $K = cZ$ for some $c \in \mathbb{C}_0$ by Result 5.13 of [56]. Since Ω and $c^{-1}\Omega$ are centrally cohomologous, the bi-Galois object N is then indeed isomorphic to the composition of the $P_1 \otimes P_2$-$M_1 \otimes M_2$-bi-Galois object $N_1 \otimes N_2$ with the canonical P_Z-$(P_1 \otimes P_2)$-bi-Galois object.

□

Then also Theorem 2.1 of [71] can be immediately adapted to yield

Corollary 9.2.3. *If $\widehat{M}$ is a generalized quantum double of $\widehat{M}_1$ and $\widehat{M}_2$, and $\hat{P}$ is comonoidally W^*-Morita equivalent with $\widehat{M}$, then $\hat{P}$ is a generalized quantum double of two von Neumann algebraic quantum groups $\hat{P}_1$ and $\hat{P}_2$ which are comonoidally W^*-Morita equivalent with respectively $\widehat{M}_1$ and $\widehat{M}_2$. Moreover, there is then a one-to-one correspondence between the P-M-bi-Galois objects and pairs of P_i-M_i-bi-Galois objects.*

Chapter 10

Application: Projective corepresentations

Let $\mathfrak{G}$ be a locally compact group, and suppose we are given a continuous map Υ of $\mathfrak{G}$ into the space of *-*automorphisms* of $B(\mathscr{H})$ for some Hilbert space $\mathscr{H}$, equipped with the point-σ-weak topology. Since any such automorphism χ is inner, i.e. of the form

$$\chi(x) = uxu^*, \qquad x \in B(\mathscr{H}),$$

for some $u \in \mathscr{U}(B(\mathscr{H}))$, the group of unitaries on $\mathscr{H}$, this means that we have a Borel covering

$$\{(g,u) \mid \mathrm{Ad(u)} = \Upsilon(g)\} \subseteq \mathfrak{G} \times \mathscr{U}(B(\mathscr{H}))$$

of $\mathfrak{G}$. When everything is separable, we can choose a Borel map $v : \mathfrak{G} \to \mathscr{U}(B(\mathscr{H}))$ which creates a section of this covering. Then v is not necessarily a *-representation, but it comes close: there exists a measurable function

$$\Omega : \mathfrak{G} \times \mathfrak{G} \to S^1,$$

with S^1 the circle group $\subseteq \mathbb{C}$, such that

$$v_{gh} = \overline{\Omega(g,h)} v_g v_h$$

(where we choose the conjugate to have compatibility with later definitions). This Ω, when interpreted as an element of $\mathscr{L}^\infty(\mathfrak{G}) \otimes \mathscr{L}^\infty(\mathfrak{G})$, will then precisely be a unitary 2-cocycle for the von Neumann algebraic quantum group $\mathscr{L}^\infty(\mathfrak{G})$. We then call the map $g \to v_g$ a unitary Ω-*representation*, and we then call *projective* unitary representation a unitary Ω-representation for *some* Ω.[1] Conversely, any unitary projective representation determines an action of $\mathfrak{G}$ on a $B(\mathscr{H})$.

This shows that there is a very close connection between actions on type I-factors and 2-cocycles. We now want to study this phenomenon for general von Neumann algebraic quantum groups. It turns out that in this case, 2-cocycles have to be replaced with general Galois objects. We then apply our results to construct a peculiar kind of comonoidal W*-Morita equivalence between a compact and a non-compact von Neumann algebraic quantum group.

Note on notation: Since in this section, we will mainly work with the non-symmetrical notion of a right Galois object, we will again follow the convention for the associated linking von Neumann algebraic quantum groupoid $\widehat{Q}$ as in section 7.3, that is: we suppress the notation for the representation on $\begin{pmatrix} \mathscr{L}^2(N) \\ \mathscr{L}^2(M) \end{pmatrix}$, while we explicitly write the notation for the standard left GNS-representation.

10.1 Projective corepresentations

Definition 10.1.1. *Let N be a right Galois object for a von Neumann algebraic quantum group M. Let $\mathscr{H}$ be a Hilbert space. A (unitary)* left N-corepresentation *for $\widehat{M}$ is a unitary $\mathcal{G} \in \hat{N} \otimes B(\mathscr{H})$ such that*

$$(\Delta_{\hat{N}} \otimes \iota)(\mathcal{G}) = \mathcal{G}_{13}\mathcal{G}_{23}.$$

If $[N]$ denotes an isomorphism class of right Galois objects for M, we call (unitary) left $[N]$-corepresentation *for $\widehat{M}$ a unitary left N-corepresentation for some $N \in [N]$.*

By a (unitary) projective corepresentation *for $\widehat{M}$, we mean a left N-corepresentation for $\widehat{M}$ for* some *right M-Galois object N.*

[1] This deviates somewhat from the commonly accepted definition, in which a projective representation is a homomorphism $\mathfrak{G} \to \mathscr{U}(B(\mathscr{H}))/S^1$. We will however make up for this by choosing an appropriate notion of isomorphism.

For any right M-Galois object N, there is a *regular left N-corepresentation* on the Hilbert space $\mathscr{L}^2(\widehat{O})$, given by the unitary $\widehat{W}^2_{21} = (J_N \otimes J_{\widehat{N}})\tilde{G}^*(J_M \otimes J_{\widehat{O}})$. In case $M = \mathscr{L}(\mathfrak{G})$ is the group von Neumann algebra of a locally compact group $\mathfrak{G}$, and N is the Ω-twisted group von Neumann algebra by a unitary 2-cocycle $\Omega \in \mathscr{L}^\infty(\mathfrak{G}) \otimes \mathscr{L}^\infty(\mathfrak{G})$, we then get back the ordinary notion of an Ω-representation. Of course, one can also easily adapt the definition to find the notion of a right N-corepresentation.

If N is a right Galois object for M, then intertwiners between two N-corepresentations $\mathcal{G}_2$ and $\mathcal{G}_1$ on respective Hilbert spaces $\mathscr{H}_2$ and $\mathscr{H}_1$ are those operators $x : \mathscr{H}_2 \to \mathscr{H}_1$ for which

$$\mathcal{G}_1(1 \otimes x) = (1 \otimes x)\mathcal{G}_2.$$

If $[N]$ is an isomorphism class of right M-Galois objects, then intertwiners between two $[N]$-corepresentations $\mathcal{G}_2$ and $\mathcal{G}_1$ on respective Hilbert spaces $\mathscr{H}_2$ and $\mathscr{H}_1$, and with respective associated right Galois objects $N_2 \in [N]$ and $N_1 \in [N]$, are those operators $x : \mathscr{H}_2 \to \mathscr{H}_1$ for which there exists an isomorphism $\Phi : N_2 \to N_1$ of right Galois objects such that

$$(\widehat{\Phi} \otimes \iota)(\mathcal{G}_1)(1 \otimes x) = (1 \otimes x)\mathcal{G}_2,$$

where $\widehat{\Phi} : \widehat{N}_1 \to \widehat{N}_2$ is the dual of Φ (the precise definition of which is easily guessed). Finally, when $\mathcal{G}_2$ and $\mathcal{G}_1$ are two projective representations with associated isomorphism classes $[N_1]$ and $[N_2]$ of right Galois objects, we define their set of intertwiners to be $\{0\}$ if $[N_1] \neq [N_2]$, and the set of $[N]$-intertwiners when $[N_1] = [N_2] = [N]$.

We then call two N-corepresentations (resp. $[N]$-corepresentations or projective corepresentations) *isomorphic* when there exists an invertible intertwiner between them. We cal an N-corepresentation (resp. $[N]$-corepresentation or projective corepresentation) *irreducible* if its intertwiners with itself are just the scalar multiples of the identity.

By the theory developed in the third section of chapter 11 (see page 340), an N-corepresentation $\mathcal{G}$ on a Hilbert space $\mathscr{H}$ is a special type of corepresentation on $\mathscr{H}$ of the associated linking von Neumann algebraic quantum groupoid $(\widehat{Q}, e)$. To be precise: given such an N-corepresentation, we give $\mathscr{H}$ the (non-faithful) $\mathbb{C}^2$-$\mathbb{C}^2$-bimodule structure with left action $a(e_i) := \delta_{i2}\, 1_{B(\mathscr{H})}$ and right action $\hat{b}(e_i) := \delta_{i1}\, 1_{B(\mathscr{H})}$, where $\{e_i \mid i \in \{1,2\}\}$ denotes the canonical basis of $\mathbb{C}^2$. Then, using the notation of section 11.3,

$$\begin{aligned} q(\mathscr{L}^2(Q)\otimes\mathscr{H}) &= (\mathscr{L}^2(O)\oplus\mathscr{L}^2(M))\otimes\mathscr{H} \\ &\subseteq \mathscr{L}^2(Q)\otimes\mathscr{H} \end{aligned}$$

and

$$\begin{aligned} q'(\mathscr{L}^2(Q)\otimes\mathscr{H}) &= (\mathscr{L}^2(P)\oplus\mathscr{L}^2(N))\otimes\mathscr{H} \\ &\subseteq \mathscr{L}^2(Q)\otimes\mathscr{H}. \end{aligned}$$

One checks that $(\pi_{\hat{Q}}\otimes\iota)(\mathcal{G})$ defines a corepresentation.

As a result of the proof of Proposition 11.3.7, the second leg of $\mathcal{G}$ will have a C*-algebra as its norm-closure (a fact which can also be proven directly). This implies in particular that any irreducible projective corepresentation will automatically be indecomposable.

Theorem 10.1.2. *Let M be a von Neumann algebraic quantum group. Then any (irreducible) projective corepresentation $\mathcal{G}$ of $\widehat{M}$ canonically gives rise to a(n ergodic) left coaction $\Upsilon = Coact(\mathcal{G})$ of $\widehat{M}$ on a type-I-factor, and any left coaction Υ on a type I-factor canonically gives rise to a left projective corepresentation $\mathcal{G} = Corep(\Upsilon)$. Moreover, $Coact\circ Corep$ is the identity, and $Corep\circ Coact$ will send a projective corepresentation to an isomorphic projective corepresentation.*

Proof. The first statement is easy: if $\mathcal{G}$ is a projective corepresentation, define

$$\Upsilon : B(\mathscr{H}) \to \widehat{M}\otimes B(\mathscr{H}) : x \to \mathcal{G}^*(1\otimes x)\mathcal{G}.$$

Then this is a coaction by the defining property of $\mathcal{G}$.

Now let $\mathscr{H}$ be a Hilbert space, and $\Upsilon : B(\mathscr{H}) \to \widehat{M}\otimes B(\mathscr{H})$ a left coaction of $\widehat{M}$. Denote by N the relative commutant of $\Upsilon(B(\mathscr{H}))$ inside $\widehat{M}\ltimes B(\mathscr{H})$. Then we have a canonical isomorphism $\Phi : \widehat{M}\ltimes B(\mathscr{H}) \to N\otimes B(\mathscr{H})$, sending $n\in N$ to $n\otimes 1$ and $\Upsilon(x)$ to $1\otimes x$ for $x\in B(\mathscr{H})$. We claim that the dual (right) coaction $\widehat{\Upsilon} : \widehat{M}\ltimes B(\mathscr{H}) \to (\widehat{M}\ltimes B(\mathscr{H}))\otimes M$ restricts to a coaction α_N of M on N. Indeed: choose an orthonormal basis ξ_i of $\mathscr{H}$, with respective matrix unit system $\{e_{ij}\}$. Then for $x\in N$, we have $x = \sum_k \Upsilon(e_{k1})x\Upsilon(e_{1k})$ in the σ-strong topology. Applying $\widehat{\Upsilon}$, we get $\widehat{\Upsilon}(x) = \sum_k(\Upsilon(e_{k1})\otimes 1)\widehat{\Upsilon}(x)(\Upsilon(e_{1k})\otimes 1)$, whose first leg clearly commutes with $\Upsilon(B(\mathscr{H}))$.

We now show that (N, α_N) is a right M-Galois object. Ergodicity is clear, since $1_N \otimes B(\mathscr{H})$ is the algebra of coinvariants for

$$\mathrm{Ad}(\Sigma)_{23}(\alpha_N \otimes \iota) = (\Phi \otimes \iota)\widehat{\Upsilon} \circ \Phi^{-1}.$$

Also integrability follows easily by this, $\widehat{\Upsilon}$ being integrable. Since we have a canonical isomorphism $(\widehat{M} \ltimes B(\mathscr{H})) \rtimes M \cong (N \rtimes M) \otimes B(\mathscr{H})$, and the first space is $\cong B(\mathscr{H}) \otimes B(\mathscr{L}^2(M))$, also $N \rtimes M$ must be a type I factor, from which it follows that the Galois homomorphism for N is necessarily an isomorphism.

We show that the original coaction is implemented by an N-corepresentation. Denote by Tr the canonical nsf trace on $B(\mathscr{H})$, by $\widehat{\mathrm{Tr}}$ the dual weight on $\widehat{M} \ltimes B(\mathscr{H})$ with respect to Tr. Then we have $\widehat{\mathrm{Tr}} = (\varphi_N \otimes \mathrm{Tr}) \circ \Phi$. Hence we obtain a unitary

$$u : \mathscr{L}^2(M) \otimes \mathscr{L}^2(B(\mathscr{H})) \to \mathscr{L}^2(N) \otimes \mathscr{L}^2(B(\mathscr{H}))$$

such that

$$\Lambda_M(m) \otimes \Lambda_{\mathrm{Tr}}(x) \to (\Lambda_N \otimes \Lambda_{\mathrm{Tr}})(\Phi(m \otimes 1)(1 \otimes x))$$

for $m \in \mathscr{N}_{\varphi_M}$ and x Hilbert-Schmidt. But identifying $\mathscr{L}^2(B(\mathscr{H}), \mathrm{Tr})$ with $\mathscr{H} \otimes \overline{\mathscr{H}}$, and observing that u is right $B(\mathscr{H})$-linear, we must have that $u = \mathcal{G} \otimes 1$ for some unitary

$$\mathcal{G} : \mathscr{L}^2(M) \otimes \mathscr{H} \to \mathscr{L}^2(N) \otimes \mathscr{H}.$$

We proceed to show that $\mathcal{G}$ is indeed an N-corepresentation implementing Υ. First of all, it is not difficult to see that $\mathcal{G} \in \hat{N} \bar{\otimes} B(\mathscr{H})$: for $m \in \mathscr{N}_{\varphi_M}$ and x Hilbert-Schmidt, and $\xi, \eta \in \mathscr{L}^2(M)$ with $\xi \in \mathscr{D}(\delta_M^{-1/2})$, we have, putting $\omega = \omega_{\xi,\eta}$ and $\omega_\delta = \omega_{\delta_M^{-1/2}\xi,\eta}$ and denoting by U the unitary implementation of α_N,

$$\begin{aligned}
&u((\iota\otimes\omega)(V_M)\otimes 1)(\Lambda_M(m)\otimes\Lambda_{\mathrm{Tr}}(x))\\
&\quad= u(\Lambda_M((\iota_M\otimes\omega_\delta)(\Delta_M(m)))\otimes\Lambda_{\mathrm{Tr}}(x))\\
&\quad= (\Lambda_N\otimes\Lambda_{\mathrm{Tr}})(\Phi(((\iota_M\otimes\omega_\delta)(\Delta_M(m)))\otimes 1)(1\otimes x))\\
&\quad= (\Lambda_N\otimes\Lambda_{\mathrm{Tr}})(\Phi((((\iota_M\otimes\omega_\delta)(\Delta_M(m)))\otimes 1)\Upsilon(x)))\\
&\quad= (\Lambda_N\otimes\Lambda_{\mathrm{Tr}})(\Phi((\iota_{\widehat{M}\ltimes B(\mathscr{H})}\otimes\omega_\delta)(\widehat{\Upsilon}((m\otimes 1)\Upsilon(x)))))\\
&\quad= (\Lambda_N\otimes\Lambda_{\mathrm{Tr}})((\iota_N\otimes\omega_\delta\otimes\iota_{B(\mathscr{H})})(\alpha_N\otimes\iota_{B(\mathscr{H})})\Phi((m\otimes 1)\Upsilon(x)))\\
&\quad= ((\iota\otimes\omega)(U)\otimes 1)(\Lambda_N\otimes\Lambda_{\mathrm{Tr}})(\Phi((m\otimes 1)\Upsilon(x)))\\
&\quad= ((\iota\otimes\omega)(U)\otimes 1)u(\Lambda_M(m)\otimes\Lambda_{\mathrm{Tr}}(x)),
\end{aligned}$$

so that $\mathcal{G}((\iota\otimes\omega)(V_M)\otimes 1) = ((\iota\otimes\omega)(U)\otimes 1)\mathcal{G}$, which is sufficient to conclude that the first leg of $\mathcal{G}$ is in $\widehat{N}$.

Also, it is easy to see that $\mathcal{G}$ implements Υ: since $u(\iota_{\widehat{M}}\otimes\pi_{B(\mathscr{H})})(\Upsilon(x)) = (1\otimes\pi_{B(\mathscr{H})}(x))u$ on $\mathscr{L}^2(M)\otimes\mathscr{L}^2(B(\mathscr{H}))$, we have $\mathcal{G}\Upsilon(x) = (1\otimes x)\mathcal{G}$ on $\mathscr{L}^2(M)\otimes\mathscr{H}$.

So the only thing left to show, is that $\mathcal{G}$ satisfies

$$(\widehat{\Delta}_{12}\otimes\iota)(\mathcal{G}) = \mathcal{G}_{13}\mathcal{G}_{23}.$$

Writing out $\widehat{\Delta}_{12}$ and tensoring by $1_{\overline{\mathscr{H}}}$ to the right, this translates into proving that $\tilde{G}^*_{12}u_{23}(W_{\widehat{M}})_{12} = u_{13}u_{23}$, with $\tilde{G}$ the Galois unitary for N. Moving $\tilde{G}$ to the other side, and multiplying to the left with Σ_{12}, this becomes $u_{13}(W_M)^*_{12}\Sigma_{12} = \Sigma_{12}\tilde{G}_{12}u_{13}u_{23}$. This identity can then again be proven using a simple matrix algebra argument: we can write $\Phi(m\otimes 1) = \sum_{i,j}\Phi_{ij}(m)\otimes e_{ij}$ with $\Phi_{ij}(m) = \sum_k\Upsilon(e_{ki})(m\otimes 1)\Upsilon(e_{jk}) \in N$, where the sums are in the σ-strong topology. Then for $m,n\in\mathscr{N}_{\varphi_M}$ and x Hilbert-Schmidt, we make the following calculation: on the one hand,

$$\begin{aligned}
&u_{13}W^*_{12}\Sigma_{12}(\Lambda_M(m)\otimes\Lambda_M(n)\otimes\Lambda_{\mathrm{Tr}}(x))\\
&\quad= u_{13}(\Lambda_M\otimes\Lambda_M\otimes\Lambda_{\mathrm{Tr}})(\Delta_M(m)(n\otimes 1)\otimes x)\\
&\quad= (\Lambda_N\otimes\Lambda_M\otimes\Lambda_{\mathrm{Tr}})(\sum_{i,j}((\Phi_{ij}\otimes\iota)(\Delta_M(m)(n\otimes 1))\otimes e_{ij}x)),
\end{aligned}$$

while on the other hand,

$$\begin{aligned}
&\Sigma_{12}\tilde{G}_{12}u_{13}u_{23}(\Lambda_M(m)\otimes\Lambda_M(n)\otimes\Lambda_{\mathrm{Tr}}(x))\\
&= \Sigma_{12}\tilde{G}_{12}u_{13}(\Lambda_M\otimes\Lambda_N\otimes\Lambda_{\mathrm{Tr}})(\sum_{i,j} m\otimes\Phi_{ij}(n)\otimes e_{ij}x)\\
&= \Sigma_{12}\tilde{G}_{12}(\Lambda_N\otimes\Lambda_N\otimes\Lambda_{\mathrm{Tr}})(\sum_{i,j,r}(\Phi_{ri}(m)\otimes\Phi_{ij}(n)\otimes e_{rj}x))\\
&= (\Lambda_N\otimes\Lambda_M\otimes\Lambda_{\mathrm{Tr}})(\sum_{i,j,r}((\alpha_N(\Phi_{ri}(m))\otimes 1)(\Phi_{ij}(n)\otimes 1\otimes e_{rj}x)))\\
&= (\Lambda_N\otimes\Lambda_M\otimes\Lambda_{\mathrm{Tr}})(\sum_{i,j,r}((\Phi_{ri}\otimes\iota)(\Delta_M(m))\otimes 1)(\Phi_{ij}(n)\otimes 1\otimes e_{rj}x))\\
&= (\Lambda_N\otimes\Lambda_M\otimes\Lambda_{\mathrm{Tr}})(\sum_{j,r}((\Phi_{rj}\otimes\iota)(\Delta_M(m)(n\otimes 1))\otimes e_{rj}x)),
\end{aligned}$$

where we have used $\sum_i \Phi_{ri}(m)\Phi_{ij}(n) = \Phi_{rj}(mn)$ for $m,n \in M$ in the last step. So we are done.

Now suppose we are given an N_1-corepresentation $\mathcal{G}_1$. Let $\mathcal{G}_2$ be the projective corepresentation (Corep $\circ$ Coact)($\mathcal{G}_1$), with associated right Galois object N_2. Then since $\mathcal{G}_1$ and $\mathcal{G}_2$ implement the same coaction on $B(\mathscr{H})$, we must have $\mathcal{G}_1\mathcal{G}_2^* = v\otimes 1$ for some unitary $v : \mathscr{L}^2(N_2) \to \mathscr{L}^2(N_1)$. Since v is a right $\widehat{M}$-module map, we can extend the (well-defined) map $\hat{Q}_{2,12} \to \hat{Q}_{1,12} : z \to vz$ to an isomorphism Ψ of the linking von Neumann algebras $\hat{Q}_2$ and $\hat{Q}_1$. From the fact that $\mathcal{G}_1$ and $\mathcal{G}_2$ are projective corepresentations, it is easy to deduce that

$$\hat{\Delta}_{1,12}(vz) = (v\otimes v)\hat{\Delta}_{2,12}(z)$$

for $z \in \hat{Q}_{2,12}$. Hence Ψ is an isomorphism of linking von Neumann algebraic quantum groupoids, keeping the right lower corner fixed. Thus N_1 and N_2 are isomorphic by a map $\hat{\Psi}$, and moreover $(\Psi\otimes\iota)(\mathcal{G}_2) = \mathcal{G}_1$.

Finally, it is trivial to see that under this correspondence, irreducible projective corepresentations correspond to ergodic coactions.

□

The following is also a generalization of a classical result.

Theorem 10.1.3. *Suppose M is a von Neumann algebraic quantum group for which M has a separable predual, and let $\mathscr{H}$ be a separable infinite-*

dimensional Hilbert space. Then there is a natural one-to-one correspondence between outer equivalence classes of coactions of $\widehat{M}$ on $B(\mathscr{H})$, and isomorphism classes of right Galois objects (with separable predual) for M.

Proof. First suppose that Υ_1 and Υ_2 are two coactions of $\widehat{M}$ on $B(\mathscr{H})$ which are outer equivalent by a unitary $v \in \widehat{M} \otimes B(\mathscr{H})$. Then we get an isomorphism

$$\Phi : \widehat{M} \underset{\Upsilon_1}{\ltimes} B(\mathscr{H}) \to \widehat{M} \underset{\Upsilon_2}{\ltimes} B(\mathscr{H}) : z \to vzv^*,$$

which obviously sends $\Upsilon_1(B(\mathscr{H}))$ to $\Upsilon_2(B(\mathscr{H}))$ (see the proof of Proposition 4.2 of [85]). Hence if N_i denotes the right M-Galois object constructed from Υ_i as in the previous Theorem, N_1 is sent to N_2 by Φ. But Φ also preserves the dual right coaction, since $(V_M)_{13}v_{12} = v_{12}(V_M)_{13}$. So $\Phi_{|N_1}$ gives an M-equivariant isomorphism from N_1 to N_2.

Conversely, suppose that N is a right M-Galois object, and that Υ_1 and Υ_2 are two left coactions of $\widehat{M}$ on $B(\mathscr{H})$, which are induced by respective N-corepresentations $\mathcal{G}_1$ and $\mathcal{G}_2$. Put $v = \mathcal{G}_2^*\mathcal{G}_1 \in \widehat{M} \otimes B(\mathscr{H})$. Then v is an Υ_1-cocycle:

$$\begin{aligned}
(\Delta_{\widehat{M}} \otimes \iota)(v) &= \mathcal{G}_{2,23}^*\mathcal{G}_{2,13}^*\mathcal{G}_{1,13}\mathcal{G}_{1,23} \\
&= v_{23}\mathcal{G}_{1,23}^*v_{13}\mathcal{G}_{1,23} \\
&= v_{23}(\iota \otimes \Upsilon_1)(v),
\end{aligned}$$

and obviously $\Upsilon_2(x) = v\Upsilon_1(x)v^*$ for $x \in B(\mathscr{H})$. Hence Υ_1 and Υ_2 are outer equivalent.

Now for any right Galois object N with separable predual, there exists a coaction on $B(\mathscr{H})$ which has N as its associated Galois object: for example, one can take $\mathscr{H} \cong \mathscr{L}^2(\hat{O}) \otimes \mathscr{H}$ and equip it with the coaction

$$\Upsilon : B(\mathscr{L}^2(\hat{O}) \otimes \mathscr{H}) \to \widehat{M} \otimes \mathscr{L}^2(\hat{O}) \otimes \mathscr{H} :$$

$$\Upsilon(x) = (\widehat{W}_{21}^2)^*(1 \otimes x)\widehat{W}_{21}^2,$$

i.e., take an amplification of the coaction coming from the regular left projective corepresentation of a right Galois object. This observation then ends the proof of the proposition. ∎

10.2 Projective representations

We now introduce the concept dual to that of a projective corepresentation. We will use the notations of section 7.6.

Definition 10.2.1. *Let M be a von Neumann algebraic quantum group, and N a right M-Galois object.* A continuous left N-representation *of M is a non-degenerate *-representation of C^u on some Hilbert space. A* continuous projective left representation *of M is a continuous left N-representation for* some *right M-Galois object N.*

We show in the following proposition that the notions of projective corepresentation and projective representation are dual to each other, and how certain properties are transported along this duality. We first introduce a definition.

Definition 10.2.2. *Let N be a right Galois object for a von Neumann algebraic quantum group M. We call an N-corepresentation $\mathcal{G}$ on a Hilbert space $\mathscr{H}$* square integrable *if the associated left coaction*

$$\Upsilon : B(\mathscr{H}) \to \widehat{M} \otimes B(\mathscr{H}) : x \to \mathcal{G}^*(1 \otimes x)\mathcal{G}$$

of $\widehat{M}$ is integrable.

Proposition 10.2.3. *Let N be a right Galois object for M.*

1. *There is a one-to-one correspondence between (irreducible) right N-corepresentations and (irreducible) continuous left N-representations.*

2. *There is a one-to-one correspondence between (irreducible) square integrable N-corepresentations and (irreducible) unital normal *- representations of O.*

Proof. The first statement is immediate by the remarks before Theorem 10.1.2 and Proposition 11.3.8. It is moreover clear that under this duality, irreducibility is preserved.

Now suppose that $\mathcal{G}$ is a *square integrable* N-corepresentation on $\mathscr{H}$. Let Υ be the associated left coaction of $\widehat{M}$ on $B(\mathscr{H})$, and put $T_\Upsilon = (\psi_{\widehat{M}} \otimes \iota_{B(\mathscr{H})})\Upsilon$. Denote by $\pi_{\mathcal{G}}$ the *-representation of $\mathscr{L}^1_*(\hat{N})$ on $\mathscr{H}$ associated to $\mathcal{G}$, and denote by $\lambda_{\widehat{N}}$ the *-representation of $\mathscr{L}^1_*(\widehat{N})$ on $\mathscr{L}^2(O)$ (so $\lambda_{\widehat{N}}(\omega) = (\omega \otimes \iota)(\widehat{W}_{21})$). Take $\xi, \eta \in \mathscr{H}$, and $x \in \mathscr{N}_{T_\Upsilon}$. Then with $\omega = \omega_{\xi, x^*\eta}$, we have

$$\psi_{\widehat{M}}((\iota \otimes \omega)(\mathcal{G})^*(\iota \otimes \omega)(\mathcal{G})) \leqslant \|\omega_{\xi,\eta}\| \cdot \psi_{\widehat{M}}((\iota \otimes |\omega_{\xi,\eta}|)(\mathcal{G}^*(1 \otimes x^*x)\mathcal{G})),$$

which is finite. Hence $(\iota \otimes \omega)(\mathcal{G}) \in \mathscr{N}_{\psi_{\hat{Q}}}$. Now one can compute that if $y \in \widehat{N} \cap \mathscr{N}_{\psi_{\hat{Q}}}$, and $\omega' \in \mathscr{L}^1_*(\widehat{N})$ is such that $R_Q(\lambda_{\widehat{N}}(\omega')) \in \mathscr{N}_{\varphi_Q}$, then $\omega'(y) = \langle \widehat{\Gamma}_{12}(y), J_Q \Lambda_N(R_Q(\lambda_{\widehat{N}}(\omega')))\rangle$ (for example, it follows purely from the defining equality of Theorem 3.10.(v) of [30], using various $^{\mathrm{op}}$-identifications). Taking such an ω', and also an arbitrary $\omega'' \in (\widehat{N})_*$, we see that $R_Q(\lambda_{\widehat{N}}(\omega' \cdot \omega'')) \in \mathscr{N}_{\varphi_Q}$, and, with ω as before,

$$\begin{aligned}
&\omega(\pi_{\mathcal{G}}(\omega') \cdot \pi_{\mathcal{G}}(\omega''))\\
&\quad = ((\omega' \cdot \omega'') \otimes \omega)(\mathcal{G})\\
&\quad = \langle \widehat{\Gamma}_{12}((\iota \otimes \omega)(\mathcal{G})), J_Q \Lambda_N(R_Q(\lambda_{\widehat{N}}(\omega' \cdot \omega'')))\rangle\\
&\quad = \langle (J_Q(R_Q(\lambda_{\widehat{N}}(\omega'')))^* J_Q)\widehat{\Gamma}_{12}((\iota \otimes \omega)(\mathcal{G})), J_Q \Lambda_N(R_Q(\lambda_{\widehat{N}}(\omega')))\rangle
\end{aligned}$$

So the functional $\omega(\pi_{\mathcal{G}}(\omega') \cdot \pi_{\mathcal{G}} \lambda_{\widehat{N}}^{-1}(\,\cdot\,))$ can be extended from a functional on $\lambda_{\widehat{N}}(\mathscr{L}^1_*(\widehat{N}))$ to a normal functional on O. Now linear combinations of functionals of the form $\omega(\pi_{\mathcal{G}}(\omega')\,\cdot\,)$ have norm-dense linear span in $B(\mathscr{H})_*$, where the fact that there are enough elements of the form ω' can be proven similarly as e.g. in Lemma 4.2 of [54]. Hence we can extend $\pi_{\mathcal{G}} \circ \lambda_{\widehat{N}}^{-1}$ from $\lambda_{\widehat{N}}(\mathscr{L}^1_*(\widehat{N}))$ to a normal representation of O.

As for the other direction, suppose π is a *-representation of $\mathscr{L}^1_*(\widehat{N})$ on a Hilbert space $\mathscr{H}$, which extends to a normal representation of O. Choose $\xi \in \mathscr{H}$. Then $\omega_{\xi,\xi} \circ \pi$ extends to a normal state on O. Hence there exists $\eta \in \mathscr{L}^2(O)$ such that $\pi(x)\xi \to x\eta$ gives a well-defined left O-linear isometry $\pi(O)\xi \to \mathscr{L}^2(O)$. Denote by p the range projection; then $p \in O'$. Since the regular left N-corepresentation $\widehat{W}^2_{21}$ is square integrable (the associated coaction of $\widehat{M}$ being a dual coaction), we get that if $y \in B(\mathscr{L}^2(O))^+$ is integrable for the associated coaction, then also pyp is integrable. So if $\mathcal{G}$ is the N-corepresentation associated to π, then, since $(\iota \otimes \pi)(\widehat{W}^2_{12}) = \mathcal{G}$, the restriction of the representation π to the closure of $\pi(N)\xi$ is square integrable. Since ξ was arbitrary, π itself will be square integrable.

□

Remark: The connection between the square integrability of a(n ordinary) unitary representation of a locally compact group and the integrability of its associated action seems first to have been noted in [68].

10.3 A counter-intuitive example

In this section, we show two suprising results. First of all, we show that there exist infinite-dimensional irreducible projective corepresentations for certain compact quantum groups, which is impossible for classical compact groups. Secondly, using this result, we show that the property of being discrete is not preserved by monoidal W*-co-Morita equivalence (which is to be contrasted with Theorem 3.8.2). We can even establish this equivalence by a *cleft* bi-Galois object. Stating this result in the dual way, this shows that one can construct a compact quantum group and a unitary 2-cocycle, in such a way that the cocycle twisted von Neumann algebraic quantum group is no longer compact. Also note that then necessarily the reduced C*-algebras underlying these quantum groups can not be the same, as one is unital and the other is not.

Proposition 10.3.1. *Let N be a bi-Galois object between von Neumann algebraic quantum groups M and P. If M is discrete, then N is a von Neumann algebraic direct sum of type I-factors. If moreover P is discrete, then the summands are finite-dimensional.*

Proof. The first assertion is easy: if M is discrete, then N, being the fixed point algebra of the dual right coaction $\widehat{\alpha_N}$ by the compact quantum group $\widehat{M}'$, must be of the above form by Lemma 5.6.5, since $\mathcal{E}_{\alpha_N} := (\iota\otimes\varphi_{\widehat{M}'})\widehat{\alpha_N}$ is a *normal* conditional expectation $N \underset{\alpha_N}{\rtimes} M \cong B(\mathscr{L}^2(N)) \rightarrow N$. We also note for further use that $\varphi_N\circ\mathcal{E}_{\alpha_N} = \mathrm{Tr}(\,\cdot\,\nabla_N)$, ∇_N the modular operator for φ_N, since by Proposition 5.7 of [85], we know that $\mathcal{E}_{\alpha_N}$ is also the operator valued weight obtained by applying the tower construction for operator valued weights to

$$\mathbb{C} = N^{\alpha_N} \underset{\varphi_N}{\subseteq} N \subseteq N \underset{\alpha_N}{\rtimes} M = B(\mathscr{L}^2(N)).$$

(See also the remark after Proposition 6.4.9.)

Now suppose also P is discrete. By Proposition 11.1.1, we know that

$$\nabla_{\varphi_Q}^{2it} = \delta_{\hat{Q}}^{-it}(J_{\hat{Q}}\delta_{\hat{Q}}^{it}J_{\hat{Q}})\delta_Q^{-it}(J_Q\delta_Q^{-it}J_Q).$$

In particular,

$$\nabla_N^{2it} = \hat{\pi}_{\gamma_N}(\delta_{\hat{P}}^{-it})\hat{\pi}'_{\alpha_N}(J_{\widehat{M}}\delta_{\widehat{M}}^{it}J_{\widehat{M}})\delta_N^{-it}(J_N\delta_N^{-it}J_N),$$

and thus

$$\nabla_N^{it} = (\delta_N^{1/2})^{-it}(J_N\delta_N^{1/2}J_N)^{it} \tag{10.1}$$

since $\widehat{M}$ and $\hat{P}$ are compact.

Let p be a minimal central projection in N. Then, since pN is a type I factor, we have a natural identification

$$pN \otimes pN' \to B(p\mathscr{L}^2(N)) : x \otimes y \to xy.$$

Under this identification, $\mathrm{Tr}(\,\cdot\, p\nabla_N)$ corresponds to the nsf weight

$$\mathrm{Tr}(\,\cdot\, p\delta_N^{-1/2}) \otimes \mathrm{Tr}(\,\cdot\, pJ_N\delta_N^{1/2}J_N)$$

by the formula (10.1), the traces being the canonical ones. On the other hand, if we write $\varphi_N(\,\cdot\, p) = \mathrm{Tr}(\,\cdot\, A)$ for some positive $A\eta pN$, then it follows easily from Lemma 5.7.10 that A^{-1} is a trace class operator, and that $\mathcal{E}_{\alpha_N}(xy) = \mathrm{Tr}(C_N(y)A^{-1})x$ for $x \in pN$ and $y \in pN'$. Now since $\mathrm{Tr}(\,\cdot\, p\nabla_N)$ also corresponds to the weight $\mathrm{Tr}(\,\cdot\, pA)\otimes\mathrm{Tr}(\,\cdot\, pJ_NA^{-1}J_N)$ on $pN\otimes pN'$, we conclude that $pJ_N\delta_N^{1/2}J_N$ is a multiple of $J_NA^{-1}J_N$, and that, in particular, $\delta_N^{1/2}$ is a trace class operator.

By looking at the inverse bi-Galois object, we conclude that also $\delta_O^{1/2} = J_{\hat{N}}\delta_N^{-1/2}J_{\hat{O}}$, and hence $\delta_N^{-1/2}$, is a trace class operator. Clearly, this is only possible if δ_N only has finitely many eigenvalues occurring with finitely many multiplicities. Hence pN is finite-dimensional.

□

Corollary 10.3.2. *Let N be a bi-Galois object between von Neumann algebraic quantum groups M and P. If M is of discrete Kac type, then P is of discrete Kac type.*

Proof. This is a direct consequence of the proof of the previous proposition, for in this case we can take $\delta_N = 1$, since $\alpha_N(\delta_N^{it}) = \delta_N^{it}\otimes\delta_M^{it}$ and $\delta_M = 1$ by assumption. So $\varphi_N = \psi_N$, and hence, recalling again Proposition 5.7 of [85], also $T_{\gamma_N} = \mathcal{E}_{\alpha_N}$, where T_{γ_N} is the nsf operator valued weight obtained by integrating out the dual coaction $\widehat{\gamma_N}$. But then $\varphi_{\hat{P}} = (T_{\gamma_N})_{|\hat{P}}$ is bounded.

Hence $\widehat{P}$ is a compact quantum group, and P discrete. Since $\gamma_N(\delta_N^{it}) = \delta_P^{it} \otimes \delta_N^{it}$, we must have $\delta_P = 1$, so P is of Kac type.

□

Remark: Note that for *compact* quantum groups, it is very well possible that a non-Kac type quantum group gets reflected in a Kac type quantum group: consider the monoidal W*-co-Morita-equivalence between the Kac type quantum group $A_o(n)$ (with $n \geqslant 3$) and the non-Kac type quantum group $SU_{-q}(2)$ with $q \in]0,1[$ such that $q + 1/q = n$ (see [10]).

Corollary 10.3.3. *Let N be a right Galois object for a* discrete *quantum group M. Choose a representative $(\mathscr{H}_i, \mathcal{G}_i)$ from each equivalence class of irreducible N-corepresentations. Then $O \cong \oplus_i B(\mathscr{H}_i)$, and we can choose the isomorphism such that $\widehat{W}_{21} \cong \oplus_i \mathcal{G}_i$. Moreover, $C^u = C = \oplus_i B_0(\mathscr{H}_i)$, where C is the associated reduced and C^u the associated universal C*-algebra of O.*

Proof. Since now *any* N-corepresentation is square integrable, Proposition 10.2.3 implies that each one comes from a normal representation of O. So the corollary follows immediately from the fact that $(1 \otimes p)\widehat{W}_{21}$ is an irreducible N-corepresentation for any minimal central projection p of O.

As for the second part: C^u will be equal to C since any representation for it factors through C. Now suppose π is a *-representation of $\frac{C}{C \cap \oplus_i B_0(\mathscr{H}_i)}$. Since

$$\frac{C}{C \cap \oplus_i B_0(\mathscr{H}_i)} \cong \frac{C + \oplus_i B_0(\mathscr{H}_i)}{\oplus_i B_0(\mathscr{H}_i)},$$

this means we also have a *-representation of $C + \oplus_i B_0(\mathscr{H}_i)$ which disappears on $\oplus_i B_0(\mathscr{H}_i)$. But the restriction of this representation to C extends to a normal representation of O, in which $\oplus_i B_0(\mathscr{H}_i)$ is σ-weakly dense. Hence this representation has be zero on O, and so $\pi = 0$.

Hence $C \subseteq \oplus_i B_0(\mathscr{H}_i)$. We now show equality. Since we know already that C consists of compact operators, we can choose a maximal family e_{jj} of orthogonal minimal projections in C, such that $\sum_i e_{jj}$ converges strictly to 1_C. If some e_{jj} were not a rank 1 projection, then we can find a non-zero projection $f \in \oplus_i B_0(\mathscr{H}_i)$ which is strictly smaller than e_{jj}. But then f can not be σ-weakly approximated by elements of C, which gives a contradiction. Hence all e_{jj} are rank 1. Now we also have that each $p_i C$ is simple: any *-representation extends to a normal representation of $B(\mathscr{H}_i)$, hence is faithful. Hence if e_{ij} are a system of matrix units in $\oplus_i B_0(\mathscr{H}_i)$ with respect to the e_{ij}, also each $e_{ij} \in e_{ii}Ce_{jj} \subseteq C$. This shows that $C = \oplus_i B_0(\mathscr{H}_i)$.

□

Combining Theorem 10.1.2, Proposition 10.3.1, Corollary 10.3.3 and Corollary 9.1.9, we obtain the following Theorem.

Proposition 10.3.4. *Let $\widehat{M}$ be a compact von Neumann algebraic quantum group. If $\widehat{M}$ admits an ergodic left coaction on an infinite-dimensional type I-factor, then there exists a von Neumann algebraic quantum group $\widehat{P}$ which is not compact but comonoidally W*-Morita equivalent with M. If moreover the underlying von Neumann algebra of $\widehat{M}$ is properly infinite, then we can take $\widehat{M} = \widehat{P}$ as von Neumann algebras.*

Note that by Proposition 7.6.2, the coaction α_N of the associated right Galois object N will be *continuous*, but by Theorem 3.8.2, it can not be *algebraic*, i.e., there is no natural dense *-algebra of N on which α_N restricts to an algebraic Galois coaction.

We now present a concrete example of a compact quantum group, with its underlying von Neumann algebra properly infinite, and admitting an ergodic coaction on an infinite-dimensional type I-factor. (I would like to thank Stefaan Vaes for help on this part.)

Let q_n be a sequence of numbers $0 < q_n \leqslant 1$. Let $F_n = \begin{pmatrix} 0 & q_n^{1/2} \\ -q_n^{-1/2} & 0 \end{pmatrix}$. Let A_n be the Hopf *-algebra underlying $SU_{q_n}(2)$. We recall that A_n is generated (as an algebra) by four elements $u_{n,ij}$, with *-structure uniquely determined by $\overline{U_n} := F_n^{-1}U_nF_n$, where $(U_n)_{ij} = u_{n,ij}$ and $(\overline{U_n})_{ij} = u^*_{n,ij}$, and with further defining relations $U_n^*U_n = I_2 = U_nU_n^*$. Let $A = \mathop{*}\limits_{n=0}^{\infty} A_n$ be the free product *-algebra of all A_n. Then A has a unital comultiplication $\Delta_A : A \to \mathop{*}\limits_{n=0}^{\infty}(A_n \odot A_n) \subseteq A \odot A$. Together with this comultiplication, A becomes a Hopf-*-algebra, the counit ε_A being the free product of the ε_{A_n}, and the antipode S_A being the free product of the S_{A_n}. Moreover, it has an invariant functional φ_A, namely the free product functional of all φ_{A_n}. So A is a *-algebraic quantum group of compact type. (We refer to [102] for details about this construction (which is made there in a slightly different way), notably Corollary 3.7 and Theorem 3.8.)

We now construct a particular coaction for A. Let $B_n = \odot_{k=0}^{n} M_2(\mathbb{C})$, and let B be the (algebraic) inductive limit by the natural inlusions $B_n \cong B_n \otimes 1 \subseteq B_{n+1}$. Interpreting U_n as an element of $M_2(\mathbb{C}) \odot A_n \subseteq M_2(\mathbb{C}) \odot A$, it becomes a unitary corepresentation of A. Denote then by $\mathcal{U}_n \in B_n \odot A$ the tensor product corepresentation of the first $n+1$ corepresentations U_k (that is, $U_0 \cdot U_1 \cdot \ldots \cdot U_n$). Then

$$\alpha_B : B \to B \odot A : x \in B_n \to \mathcal{U}_n(x \otimes 1)\mathcal{U}_n^*$$

is easily seen to be a well-defined coaction of A on B.

We now construct an α_B-invariant state ω_B on B. Let $c_q = \frac{1}{Tr(F_n^* F_n)} F_n^* F_n$, with F_n as above. We remark that c_q then has $\frac{q^2}{1+q^2}$ as its smallest eigenvalue. Let ω_n be the state $\mathrm{Tr}(c_q \cdot)$ on $M_2(\mathbb{C})$. Then it is well-known (and easy to calculate) that ω_n will be invariant for the coaction

$$\alpha_n : M_2(\mathbb{C}) \to M_2(\mathbb{C}) \odot A_n : x \to U_n(x \otimes 1)U_n^*.$$

Now put

$$\omega_B : B \to \mathbb{C} : x \in B_n \to (\bigotimes_{k=0}^{n} \omega_k)(x).$$

Then ω_B is indeed α_B-invariant, and moreover, gives a positive, unital map on B. We further remark that $(\iota \otimes \varphi_A)\alpha_B = \omega_B$, which easily follows from $\omega_n = (\iota \otimes \varphi_{A_n})\alpha_n$ and the way in which a free product functional has to be evaluated in products.

Let $(\mathscr{L}^2(B,\omega), \Lambda_\omega, \pi_\omega)$ be the GNS construction of B with respect to ω, and put $Y = \pi_\omega(B)''$. Put ω_Y the extension of ω_B to a normal state on Y. Let $\widehat{M}$ be the von Neumann algebraic quantum group associated to A. Since

$$U : \Lambda_\omega(B) \odot \Lambda_{\widehat{M}}(A) \to \Lambda_\omega(B) \odot \Lambda_{\widehat{M}}(A) :$$

$$\Lambda_\omega(b) \odot \Lambda_{\widehat{M}}(a) \to (\Lambda_\omega \odot \Lambda_{\widehat{M}})(\alpha_B(b)(1 \otimes a))$$

is a unitary map by the α_B-invariance of ω, we can extend U to a unitary

$$U : \mathscr{L}^2(B,\omega) \otimes \mathscr{L}^2(\widehat{M}) \to \mathscr{L}^2(B) \otimes \mathscr{L}^2(A).$$

Since $U(b \otimes 1)U^* = \alpha_B(b)$ for $b \in B$, it is clear that we can define a coaction Υ_Y on Y by putting

$$\Upsilon_Y : Y \to Y \otimes \widehat{M} : x \to U(x \otimes 1)U^*.$$

Since $(\iota\otimes\varphi_A)\alpha_B = \omega_B$, we also have $(\iota\otimes\varphi_{\widehat{M}})\Upsilon_Y = \omega_Y$. It follows that Υ_Y is necessarily an ergodic coaction.

To end, we choose the q_n in such a way that $\widehat{M}$ is a type III factor, and Y a type I factor. Choose $q_0 = q_1 = 1$. Then $\widehat{M}$ will be an infinite factor by Barnetts Theorem ([7], Theorem 2), since $\widehat{M}$ is isomorphic to $(\mathscr{L}^\infty[0,1],\mu) * ((\mathscr{L}^\infty[0,1],\mu) * (M_3,\mu'))$ for some von Neumann algebra M_3 and non-tracial faithful state μ' on it, with μ the ordinary Lebesgue measure. On the other hand, let the q_n go exponentially fast to zero at infinity. Then, by the remark concerning the eigenvalues of the ω_n, we will have Y will be an infinite type I factor, by the convergence rate of the q_n (see e.g. Lemma 2.14 of [3]). Hence we are done.

Remark: The compact quantum group used in the preceding example is rather big. For example, it is not a compact *matrix* quantum group, since the underlying C*-algebra is not finitely generated. It would therefore be interesting to see if one can also produce an example where a compact matrix quantum group (which are to be seen as compact quantum Lie groups) gets deformed into a non-compact one by twisting.

Chapter 11

Measured quantum groupoids on a finite basis

In this chapter, we study a special class of measured quantum groupoids (cf. [59]), namely those which have a finite-dimensional basis (i.e. with a finite underlying 'quantum set of objects'). Although we will only need the results in the special case where the base algebra is $\mathbb{C}^2$, it seemed pointless not to develop the theory in somewhat more generality. Although we have decided only to treat the case where the given weight on the base algebra is a trace, all results also hold true in the general case, with minor modifications. We will then develop an alternative definition for these measured quantum groupoids (in terms of so-called weak Hopf-von Neumann algebras), and consider their *naturally* associated C*-algebraic structures.

Remark: the notation used here is adapted to the one of [30]. Therefore, there will be some overlap with symbols used in a different context at other places. This should not lead to any confusion, as this chapter is fairly independent of the preceding ones.

11.1 Weak Hopf-von Neumann algebras

Let $(L, Q, d, f, \Gamma, T, T', \nu)$ be a measured quantum groupoid in the sense of Definition 3.7 of [30]. This means the following:

1. L, Q are von Neumann algebras,

2. d is a faithful normal unital *-homomorphism $L \to Q$,

3. f is a faithful normal unital *-anti-homomorphism $L \to Q$,
4. Γ is a faithful normal unital *-homomorphism $Q \to Q \underset{L}{{}_f*_d} Q$,
5. T is an nsf operator valued weight $Q^+ \to (d(L))^{+,\mathrm{ext}}$,
6. T' an nsf operator valued weight $Q^+ \to (f(L))^{+,\mathrm{ext}}$, and finally
7. ν is an nsf weight on L^+.

These have to satisfy the following conditions:

1. The range of d commutes elementwise with the range of f: $d(x)f(y) = f(y)d(x)$ for all $x, y \in L$,
2. $\Gamma(d(x)) = d(x) \underset{L}{{}_f\otimes_d} 1$,
3. $\Gamma(f(x)) = 1 \underset{L}{{}_f\otimes_d} f(x)$,
4. $(\Gamma \underset{L}{{}_f*_d} \iota) \circ \Gamma = (\iota \underset{L}{{}_f*_d} \Gamma) \circ \Gamma$,
5. $(\iota \underset{L}{{}_f*_d} T)\Gamma(x) = T(x) \underset{L}{{}_f\otimes_d} 1$ for $x \in \mathscr{M}_T^+$,
6. $(T' \underset{L}{{}_f*_d} \iota)\Gamma(x) = 1 \underset{L}{{}_f\otimes_d} T'(x)$ for $x \in \mathscr{M}_{T'}^+$,
7. With $\varphi = \nu \circ d^{-1} \circ T$ and $\psi = \nu \circ f^{-1} \circ T'$, the modular automorphisms σ_t^φ and σ_s^ψ commute for all $s, t \in \mathbb{R}$.

Note that the second and third condition make sense by the first one, which also endows $Q \underset{L}{{}_f*_d} Q$ with natural (anti-)embeddings of L. Then the fourth condition makes sense by the second and third. Also note that the fifth and sixth condition are equivalent with $(\iota \underset{L}{{}_f*_d} \varphi)(\Gamma(x)) = T(x)$ for $x \in \mathscr{M}_T^+$, resp. $(\psi \underset{L}{{}_f*_d} \iota)(\Gamma(x)) = T'(x)$ for $x \in \mathscr{M}_{T'}^+$ (see Definition 3.5 of [30]).

When ν is an nsf weight satisfying the final condition above, it is called *relatively invariant* w.r.t. T and T' (Definition 3.7 of [30]). In [58], the first article concerning measured quantum groupoids, another definition was given: the only difference is that there ν has to be *quasi-invariant* with respect to T and T': this means that one should have $\sigma_t^T(f(x)) = f(\sigma_{-t}^\nu(x))$ and $\sigma_t^{T'}(d(x)) = d(\sigma_t^\nu(x))$ for $x \in L$. In general, this makes this second notion of a measured quantum groupoid (which is now also called an *adapted* measured quantum groupoid) *stronger* than the first notion, as proven in [59] and the appendix of [30]. However, in the weaker theory, one can obtain a unitary antipode $R : Q \to Q$ (cf. Theorem 3.8.(i) of [30]), which will be a coinvolution in the sense of Definition 3.3 of [58]. Then if we replace T' by $R \circ T \circ R$, it will still satisfy the weak definition, and it will already satisfy the strong definition (i.e. be adapted) *if* we know only that $\sigma_t^T(f(x)) = f(\sigma_{-t}^{\nu}(x))$ for $x \in L$ (see Remark 4.3 of [58]).

The theory of measured quantum groupoids can then be developed parallel to the theory of von Neumann algebraic quantum groups. In particular, one has an antipode and a modular element, while the scaling constant is now replaced by a scaling operator. One also has a well-behaving duality theory. We refer to the preliminary sections of [30] for an overview of the precise results.

At some point, we needed the following relation between the structural operators of a measured quantum groupoid (compare [90]).

Proposition 11.1.1. *Let $(L, Q, d, f, \Gamma, T, T', \nu)$ be a measured quantum groupoid, with $T' = R \circ T \circ R$. Then*

$$\nabla_{\varphi_Q}^{2it} = \delta_{\widehat{Q}}^{-it}(J_{\widehat{Q}}\delta_{\widehat{Q}}^{it}J_{\widehat{Q}})\delta_Q^{-it}(J_Q\delta_Q^{-it}J_Q).$$

Proof. The proof is completely the same as in Proposition 3.4 of [90], using the commutation relations in Theorem 3.10 of [30] and biduality. □

We will from now on be interested in those measured quantum groupoids $(L, Q, d, f, \Gamma, T, T', \nu)$ for which L is finite-dimensional, and *we will assume this is satisfied for the rest of this chapter*. Then if we have a measured quantum groupoid, it follows from Theorem 3.8 of [30] that there exists a one-parametergroup γ_t of automorphisms on L such that $\sigma_t^T(f(x)) = f(\gamma_t(x))$ for $x \in L$. It is easy to see that any such one-parametergroup must be of the form σ_{-t}^{ϵ} for some faithful positive functional ϵ. So if we choose ϵ instead of ν, then we will have an adapted measured quantum groupoid. Moreover, it follows from Proposition 5.41 in [58] that in our case, we can always choose T in such a way that the above one-parametergroup γ_t is trivial, so that we can take for ϵ an arbitrary faithful positive trace.[1] Note that the antipode

[1] It would also be possible to continue working with arbitrary faithful positive ϵ, making the necessary adaptations here and there, but we have restricted ourselves to this (slightly simpler) case.

S, being defined with the aid of T and ϵ, may change by these alterations. This is not so bad, if we only consider (L, Q, d, f, Γ) (which could be called a *measurable* quantum groupoid): this could be seen as the von Neumann algebraic counterpart of either Hopf algebroids in the sense of [60] (whose antipode depends on some non-canonical section of a fibre product into an ordinary tensor product), the slightly more general Hopf algebroids proposed in [13] (whose antipode is indeed not unique), or of the still more general $\times_R$-Hopf algebras ([75]), which even do not carry an antipode. We will give a little further discussion concerning this point a bit later on. (Compare also the discussion at the end of section 1.2.1.)

Since L is a now a finite-dimensional C*-algebra, we can write

$$L = \oplus_{l=1}^{k} M_{n_l}(\mathbb{C}).$$

We will from now on further assume that $\epsilon = \oplus_{l=1}^{k} n_l \cdot Tr$, where Tr denotes the non-normalized trace (so the trace of a minimal projection is 1), i.e., ϵ is the non-normalized Markov trace on L. Put $\mathscr{H} = \mathscr{L}^2(Q)$. Then it follows from a straightforward computation that the map

$$v : \mathscr{H} \otimes \mathscr{H} \to \mathscr{H} \underset{\epsilon}{{}_f\otimes_d} \mathscr{H} : \xi \otimes \eta \to \xi \underset{\epsilon}{{}_f\otimes_d} \eta$$

is a coisometry (where it is easy to see that *any* $\xi \in \mathscr{H}$ is left and right bounded). It also follows readily that we have an isomorphism

$$Q \underset{L}{{}_f*_d} Q \to p(Q \otimes Q)p : x \to v^* x v,$$

where $p = v^*v \in Q \otimes Q$. Denote by Δ the (non-unital) *-homomorphism

$$\Delta : Q \to Q \otimes Q : x \to v^*\Gamma(x)v.$$

Then the coassociativity of Γ gives us the coassociativity of Δ, once we realize that $(\omega \underset{\epsilon}{{}_f*_d} \iota)\Gamma(x)$ is well-defined and equal to $(\omega \otimes \iota)\Delta(x)$ for all $x \in Q$ and $\omega \in Q_*$, by definition of the slice map (and similarly on the other side). We note then that

$$\Delta(d(x)) = (d(x) \otimes 1)\Delta(1) = \Delta(1)(d(x) \otimes 1)$$

and

$$\Delta(f(x)) = (1 \otimes f(x))\Delta(1) = \Delta(1)(1 \otimes f(x))$$

for $x \in L$, since v is an L-L-bimodule map (for the obvious L-L-bimodule structure in terms of d and f).

We can also look at the natural map

$$u : \mathscr{H} \otimes \mathscr{H} \to \mathscr{H} \underset{\epsilon^{op}}{{}_d\otimes_{\hat{f}}} \mathscr{H},$$

where $\hat{f}(x) = J_Q d(x)^* J_Q$. This will again be a coisometry.

Denote by $\mathscr{W}$ the *regular pseudo-multiplicative unitary* for the measured quantum groupoid $(L, Q, d, f, \Gamma, T, T', \epsilon)$: it is the unitary map $\mathscr{H} \underset{\epsilon}{{}_f\otimes_d} \mathscr{H} \to \mathscr{H} \underset{\epsilon^{\mathrm{op}}}{{}_d\otimes_{\hat{f}}} \mathscr{H}$ which is determined by

$$(l_\eta^{f,\epsilon})^* \mathscr{W}^* l_\xi^{d,\epsilon^{\mathrm{op}}} \Lambda_\varphi(x) = \Lambda_\varphi((\omega_{\xi,\eta} \underset{N}{{}_f*_d} \iota)\Gamma(x)), \qquad \xi, \eta \in \mathscr{H}, x \in \mathscr{N}_\varphi,$$

where $l_\xi^{d,\epsilon^{\mathrm{op}}}$ for example is the operator

$$\mathscr{H} \to \mathscr{H} \underset{\epsilon^{\mathrm{op}}}{{}_d\otimes_{\hat{f}}} \mathscr{H} : \zeta \to \xi \underset{\epsilon^{\mathrm{op}}}{{}_d\otimes_{\hat{f}}} \zeta$$

(see Theorem 3.10.(ii) of [30]). Denote $W = v^* \mathscr{W} u$. Then by Theorem 3.6 of [30], we can conclude that $W^*(1 \otimes y)W = \Delta(y)$ for $y \in Q$. By the stated defining property of $\mathscr{W}$, we see that we can define W^* directly as the map

$$\mathscr{H} \otimes \mathscr{H} \to \mathscr{H} \otimes \mathscr{H} : \Lambda_\varphi(x) \otimes \Lambda_\varphi(y) \to (\Lambda_\varphi \otimes \Lambda_\varphi)(\Delta(y)(x \otimes 1)), \qquad x, y \in \mathscr{N}_\varphi,$$

just as in the case of von Neumann algebraic quantum groups. We call the map W the *left regular multiplicative partial isometry* associated with the measured quantum groupoid.

Remark: An abstract theory of multiplicative partial isometries (in the finite-dimensional setting) was developed in [12] (see also [91]). We show further on that the defining properties of such m.p.i. are also satisfied for our W.

If we apply similar constructions to the dual $\hat{Q}$ of Q (see Theorem 3.10 of [30]), we obtain a partial isometry $\widehat{W}$. By that same theorem, we find that $\widehat{W} = \Sigma W^* \Sigma$. Denoting then by $\hat{\Delta}$ the corresponding comultiplication $\hat{Q} \to \hat{Q} \otimes \hat{Q}$, we get

$$W^* W = \Delta(1) = \sum_{i,j,l} n_l^{-1} f(e_{ji}^l) \otimes d(e_{ij}^l),$$

$$WW^* = \widehat{\Delta}^{\text{op}}(1) = \sum_{i,j,l} n_l^{-1} d(e_{ij}^l) \otimes \widehat{f}(e_{ji}^l).$$

We further denote $\widehat{\varphi} = \epsilon \circ d^{-1} \circ \widehat{T}$ and $\widehat{\psi} = \epsilon \circ \widehat{f}^{-1} \circ \widehat{T}'$, where $\widehat{T}$ and $\widehat{T}' = \widehat{R} \circ \widehat{T} \circ \widehat{R}$ are the dual operator valued weights on $\widehat{Q}$, as introduced in Theorem 3.10 of [30].

We state the commutation relations between W, d and f.

Lemma 11.1.2. *If $x \in L$, then*

1. $W(d(x) \otimes 1) = (1 \otimes d(x))W$,
2. $W(1 \otimes f(x)) = (1 \otimes f(x))W$,
3. $W(\widehat{f}(x) \otimes 1) = (\widehat{f}(x) \otimes 1)W$,
4. $W(1 \otimes \widehat{f}(x)) = (f(x) \otimes 1)W$.

Proof. These formulas follow straightforwardly by the identities in Definition 3.2.(i) and Theorem 3.6.(ii) of [30]. □

We state separately the commutation relations between $\Delta(1)$, $\widehat{\Delta}^{\text{op}}(1)$ and W:

Lemma 11.1.3.

1. $W_{13}(1 \otimes \widehat{\Delta}^{op}(1)) = (\Delta(1) \otimes 1)W_{13}$,
2. $(1 \otimes \widehat{\Delta}^{op}(1))W_{12} = W_{12}\widehat{\Delta}^{op}(1)_{13}$,
3. $\Delta(1)_{13}W_{23} = W_{23}(\Delta(1) \otimes 1)$,
4. $(1 \otimes \Delta(1))W_{12} = W_{12}(1 \otimes \Delta(1))$.

Proof. This follows by the previous lemma and the concrete form of $\Delta(1)$ and $\widehat{\Delta}^{\text{op}}(1)$ in terms of d, f and $\widehat{f}$. □

The following lemma gives some more commutation relations.

Lemma 11.1.4. *For $x \in L$, we have*

1. $W(f(x) \otimes 1) = W(1 \otimes d(x))$,
2. $(1 \otimes \widehat{f}(x))W = (d(x) \otimes 1)W$,
3. $\Delta(1)(f(x) \otimes 1) = \Delta(1)(1 \otimes d(x))$.

Proof. Choose $\xi, \eta \in \mathscr{H}$ and $x \in L$. Then

$$\begin{aligned} v(f(x) \otimes 1)(\xi \otimes \eta) &= (f(x)\xi) \;{}_f\!\underset{\epsilon}{\otimes}_d\; \eta \\ &= \xi \;{}_f\!\underset{\epsilon}{\otimes}_d\; (d(x)\eta) \\ &= v(1 \otimes d(x))(\xi \otimes \eta), \end{aligned}$$

so applying v^* to the left, we get $\Delta(1)(f(x) \otimes 1) = \Delta(1)(1 \otimes d(x))$. Multiplying to the left with W, we get the first relation. Considering the first relation for the dual, and using that $\widehat{W} = \Sigma W^* \Sigma$, we arrive at the second relation.

□

For multiplicative partial isometries, several (non-equivalent!) pentagon equations hold.

Lemma 11.1.5. *The operator W belongs to $Q \otimes \hat{Q}$, with $\{(\iota \otimes \omega)(W) \mid \omega \in B(\mathscr{H})_*\}$ σ-weakly dense in Q, and $\{(\omega \otimes \iota)(W) \mid \omega \in B(\mathscr{H})\}_*$ σ-weakly dense in $\hat{Q}$. Moreover, the following equations hold:*

1. $W_{12}W_{13}W_{23} = W_{23}W_{12}$,
2. $W_{12}^*W_{23}W_{12} = W_{13}W_{23}$,
3. $W_{23}W_{12}W_{23}^* = W_{12}W_{13}$.

Proof. The first statements follow straightforwardly by the corresponding results for $\mathscr{W}$ in Theorem 3.6 and Theorem 3.10 of [30].

Now note that for any faithful positive $\omega_1, \omega_2 \in L_* = L^*$, there exist $c_1, c_2 \in R_0^+$ with $c_1\omega_1 \leqslant \omega_2 \leqslant c_2\omega_2$. So the space $Q_*^{d,f}$ appearing in Theorem 3.10.(ii) of [30] is just Q_* itself. Now choose $\omega_1, \omega_2 \in Q_*$. Then, using the notation and results of that same Theorem 3.10.(ii), we get

$$\begin{aligned} (\omega_1 \otimes \omega_2 \otimes \iota)((\Delta \otimes \iota)(W)) &= (((\omega_1 \;{}_f\!\underset{\epsilon}{*}_d\; \omega_2) \circ \Gamma) \otimes \iota)(W) \\ &= \hat{\pi}(\omega_1 \cdot \omega_2) \\ &= \hat{\pi}(\omega_1)\hat{\pi}(\omega_2) \\ &= (\omega_1 \otimes \omega_2 \otimes \iota)(W_{13}W_{23}), \end{aligned}$$

and hence $(\Delta \otimes \iota)(W) = W_{13}W_{23}$. Then because W implements the comultiplication, we get the second identity of the statement. The third identity follows by a similar reasoning for the dual.

Finally, we have

$$\begin{aligned} W_{23}W_{12} &= W_{23}(1\otimes\Delta(1))W_{12} \\ &= W_{23}W_{12}(1\otimes\Delta(1)) \\ &= W_{23}W_{12}W_{23}^*W_{23} \\ &= W_{12}W_{13}W_{23}, \end{aligned}$$

where we used the fourth identity of Lemma 11.1.3 in the second step, and the third identity of this lemma in the last step. □

We now want to give a description of measured quantum groupoids which avoids the use of fibre products. We warn however that this new description is not very elegant, and that we do not really know how to see it as a specific case of a more general theory of 'weak Hopf-von Neumann algebras (with integrals)'.

Definition 11.1.6. *Let $L = \oplus_{l=1}^k M_{n_l}(\mathbb{C})$ be a finite-dimensional C^*-algebra, Q a von Neumann algebra, and d (resp. f) a faithful unital *-homomorphism (resp. anti-*-homomorphism) from L into Q. Let $\Delta : Q \to Q\otimes Q$ be a (not necessarily unital) faithful normal *-homomorphism satisfying the coassociativity condition. Assume further that f and d have pointwise commuting ranges, that $\Delta(d(x)) = (d(x)\otimes 1)\Delta(1)$, that $\Delta(f(x)) = (1\otimes f(x))\Delta(1)$, and that*

$$\Delta(1) = \sum_{i,j,l} n_l^{-1} f(e_{l,ji})\otimes d(e_{l,ij}).$$

Then we call (L, Q, d, f, Δ) a weak Hopf-von Neumann algebra with finite basis.

The origin for this terminology is to be found in the theory of weak Hopf C*-algebras, developed in [11] (see also our first chapter). Also, as for Hopf-von Neumann algebras, the name is not very suitable[2], since there is no notion of antipode around, but the connection with the theory of Hopf-von

[2] In fact, this definition is adapted to the canonical (non-normalized) Markov trace on the base space L. Allowing general positive functionals $\epsilon = \oplus_{l=1}^k \mathrm{Tr}(\,\cdot\, F_l)$, where each F_l is an invertible positive matrix, we should only ask that $\Delta(1) = \sum_{i,j,l} C_l^{-1} F_{l,i}^{-1/2} F_{l,j}^{-1/2} f(e_{ji}^l)\otimes d(e_{ij}^l)$, where the e_{ij}^l are matrix units for which $F_l = \sum_i F_{l,i} e_{ii}^l$, and where $C_l = \sum_{i=1}^{n_l} F_{i,l}^{-1}$. This would then have to be the most general definition for 'a weak Hopf-von Neumann algebra with finite basis'.

Neumann algebras is immediate.

It should be clear, by the preceding discussion, that for a weak Hopf-von Neumann algebra, we have

$$\Delta(1)(Q \otimes Q)\Delta(1) \cong Q \underset{L}{{}_f *_d} Q,$$

which is spatially implemented by the unitary

$$\Delta(1)(\mathscr{L}^2(Q) \otimes \mathscr{L}^2(Q)) \to \mathscr{L}^2(Q) \underset{\epsilon}{{}_f\otimes_d} \mathscr{L}^2(Q) :$$

$$\xi \otimes \eta \to \xi \underset{\epsilon}{{}_f\otimes_d} \eta,$$

where ϵ still denotes the non-normalized Markov trace. Then denoting by Γ the map Δ composed with this isomorphism, we see that (L, Q, d, f, Γ) satisfies all requirements of a measured quantum groupoid which do not mention (operator valued) weights (i.e., is a Hopf bimodule in the sense of [34]).

Definition 11.1.7. *Let (L, Q, d, f, Δ) be a weak Hopf-von Neumann algebra with finite basis. We say that (L, Q, d, f, Δ)* admits integrals *when there exists an nsf operator-valued weight T from Q to $d(L)$ and an nsf operator-valued weight T' from Q to $f(L)$, such that, denoting $\varphi = \epsilon \circ d^{-1} \circ T$ and $\psi = \epsilon \circ f^{-1} \circ T'$, we have*

$$\varphi((\omega \otimes \iota)\Delta(x)) = \omega(T(x)) \qquad \textit{for all } x \in \mathscr{M}_T^+, \omega \in Q_*^+,$$

$$\psi((\iota \otimes \omega)\Delta(x)) = \omega(T'(x)) \qquad \textit{for all } x \in \mathscr{M}_{T'}^+, \omega \in Q_*^+,$$

and such that σ_t^T (resp. $\sigma_t^{T'}$) leave $d(L)$ (resp. $f(L)$) pointwise invariant. We then call the septuple $(L, Q, d, f, \Delta, T, T')$ a weak Hopf-von Neumann algebra with finite basis and integrals.

Then it is easy to see that weak Hopf-von Neumann algebras with finite basis and integrals correspond precisely to those measured quantum groupoids with finite basis which are adapted with respect to the (non-normalized) Markov trace on L. We can hence also speak about the left regular multiplicative partial isometry for such a weak Hopf-von Neumann algebra.

Remark: If we would have allowed ϵ to be arbitrary, we would have that the weak Hopf C*-algebras of [11] correspond precisely to the *finite dimensional* weak Hopf-von Neumann algebras with finite basis and integrals. With ϵ the canonical Markov trace, we get back the weak Hopf C*-algebra whose antipode squared restricts to the identity on $d(L)$. Finally, when we also ask that $\varphi = \epsilon \circ T$ is a trace, we get back the (finite dimensional) generalized Kac algebras from [106].

It is handy to have a stronger form of left invariance around concerning weak Hopf-von Neumann algebras with finite basis and integrals.

Lemma 11.1.8. *Let $(L, Q, d, f, \Gamma, T, T')$ be a weak Hopf-von Neumann algebra with finite basis and integrals. Then for $x \in Q^+$, we have*

$$T(x) = (\iota \otimes \varphi)\Delta(x).$$

Proof. This follows straightforwardly from Theorem 4.12 of [30].

□

Example 11.1.9. *Linking and co-linking von Neumann algebraic quantum groupoids are weak Hopf-von Neumann algebras with finite basis and integrals.*

Proof. Let $(\widehat{Q}, e, \Delta_{\widehat{Q}})$ be a linking von Neumann algebraic quantum groupoid. Define $L = \mathbb{C}^2$, and $d = f$ the map

$$\mathbb{C}^2 \to \widehat{Q} : (a, b) \to a(1_{\widehat{Q}} - e) + be.$$

The non-normalized Markov trace on L is simply the functional

$$\epsilon : \mathbb{C}^2 \to \mathbb{C} : (a, b) \to a + b.$$

Let T (resp. T') be the unique nsf operator valued weight $\widehat{Q} \to d(L)$ such that $\epsilon \circ d^{-1} \circ T = \varphi_{\widehat{P}} \oplus \varphi_{\widehat{M}}$ (resp. $\epsilon \circ f^{-1} \circ T' = \psi_{\widehat{P}} \oplus \psi_{\widehat{M}}$). Then it is immediate that $(\mathbb{C}^2, \widehat{Q}, d, f, \Delta_{\widehat{Q}}, T, T')$ will be a weak Hopf-von Neumann algebra with finite basis and integrals.

We also substantiate the claim here, made at page 244, that linking von Neumann algebraic quantum groupoids are precisely those measured quantum groupoids with base $\mathbb{C}^2$ and coinciding source and target map with range outside the center of the measured quantum groupoid. Let

$$(\mathbb{C}^2, \widehat{Q}, d, d, \Gamma_{\widehat{Q}}, T, T', \nu)$$

be such a measured quantum groupoid. Then we can change ν into the functional ϵ from the previous paragraph, without changing the further structure (since $\mathbb{C}^2$ has no non-trivial one-parameter automorphism groups). Thus we can work with the associated weak Hopf von Neumann algebra with integrals $(\mathbb{C}^2, \widehat{Q}, d, d, \Delta_{\widehat{Q}}, T, T')$. Denote $e := d(\widehat{Q})$. The fact that T is an operator valued weight on $d(\mathbb{C}^2)$ implies that it is of the form

$$T(x) = \varphi_{\widehat{P}}((1_{\widehat{Q}} - e)x(1_{\widehat{Q}} - e)) + \varphi_{\widehat{M}}(exe), \qquad x \in \mathscr{M}_T^+,$$

for certain nsf weights $\varphi_{\widehat{P}}$ and $\varphi_{\widehat{M}}$ on resp. $\widehat{P} = (1_{\widehat{Q}} - e)\widehat{Q}(1_{\widehat{Q}} - e)$ and $\widehat{M} = e\widehat{Q}e$, and similarly for T', giving weights $\psi_{\widehat{P}}$ and $\psi_{\widehat{M}}$ on resp. $\widehat{P}$ and $\widehat{M}$. Since $\Delta_{\widehat{Q}}(\widehat{M}) \subseteq \widehat{M} \otimes \widehat{M}$, and similarly for $\widehat{P}$, it is easily verified that $\widehat{M}$ and $\widehat{P}$ are von Neumann algebraic quantum groups, the invariant weights being provided by the constituents of T and T'. Proposition 7.4.3 then lets us conclude that indeed $(\widehat{Q}, e, \Delta_{\widehat{Q}})$ is a linking von Neumann algebraic quantum groupoid.

Now let $(Q, \{p_{ij}\}, \Delta_Q)$ be a *co*-linking von Neumann algebraic quantum groupoid. Again take L and ϵ the same as in the first paragraph, but now define

$$d : \mathbb{C}^2 \to Q : (a, b) \to ap_{11} + bp_{21} + ap_{12} + bp_{22},$$
$$\widehat{f} : \mathbb{C}^2 \to Q : (a, b) \to ap_{11} + ap_{21} + bp_{12} + bp_{22}.$$

Define

$$T : Q^+ \to d([0, +\infty] \times [0, +\infty]) : x_{ij} \to \varphi_{ij}(x_{ij})(p_{i1} + p_{i2})$$

and

$$T' : Q^+ \to \widehat{f}([0, +\infty] \times [0, +\infty]) : x_{ij} \to \psi_{ij}(x_{ij})(p_{1j} + p_{2j}).$$

The invariance properties of the φ_{ij} and ψ_{ij} then easily give that T and T' satisfy the invariance properties necessary to make $(\mathbb{C}^2, Q, d, \widehat{f}, \Delta_Q, T, T')$ a weak Hopf-von Neumann algebra with finite basis and integrals.

We now substantiate the claims made on page 246. First, assume given a measured quantum groupoid $(\mathbb{C}^2, Q, d, \widehat{f}, \Gamma_Q, T, T', \nu)$ for which d and f have range in the center $\mathscr{Z}(Q)$ of Q and generate a copy of $\mathbb{C}^4$. We can again suppose that $\nu = \epsilon$, so that we can work with the associated weak Hopf-von Neumann algebra with finite basis and integrals $(\mathbb{C}^2, Q, d, \widehat{f}, \Delta_Q, T, T')$.

Then defining $p_{ij} = d(e_i)\widehat{f}(e_j)$, the p_{ij} obviously satisfy the conditions with respect to Δ_Q as in the definition of a co-linking von Neumann algebraic quantum groupoid. Then there also exist nsf weights φ_{ij} and ψ_{ij} on Q_{ij} such that T and T' take the form as in the previous paragraph. The invariance properties of φ_Q and ψ_Q then easily give the invariance properties necessary to make $(Q, \{p_{ij}\}, \Delta_Q)$ a co-linking von Neumann algebraic quantum groupoid.

Finally, we want to show that co-linking von Neumann algebraic quantum groupoids are precisely the duals of the linking von Neumann algebraic quantum groupoids. But this is easy: a measured quantum groupoid $(\mathbb{C}^2, \widehat{Q}, d, f, \Gamma_{\widehat{Q}}, T, T', \nu)$ satisfies $f = d$ and 'range of d not in the center of $\widehat{Q}$' iff its dual $(\mathbb{C}^2, Q, d, \widehat{f}, \Gamma_Q, T, T', \epsilon)$ satisfies 'd and $\widehat{f}$ have range in the center of Q and generate a copy of $\mathbb{C}^4$', which follows immediately from the way in which the dual is defined.

□

11.2 Reduced weak Hopf C*-algebras

Let $(L, Q, d, f, \Gamma, T, T', \epsilon)$ be an adapted measured quantum groupoid with L finite-dimensional and ϵ the canonical non-normalized Markov trace as in the previous section. We keep using the same notations. In particular, we still write $\mathscr{H} = \mathscr{L}^2(Q)$, and W denotes the associated left regular multiplicative partial isometry.

Denote by D the normclosure of $\{(\iota \otimes \omega)(W) \mid \omega \in \widehat{Q}_*\}$.

Proposition 11.2.1. *The Banach space D is a C*-algebra of operators.*

Proof. From the third pentagon equation in Lemma 11.1.5, and the fact that $\widehat{W} = \Sigma W^* \Sigma$, we get that

$$(\iota \otimes \omega_1)(W) \cdot (\iota \otimes \omega_2)(W) = (\iota \otimes (\omega_1 \otimes \omega_2)\widehat{\Delta}^{\mathrm{op}})(W)$$

for $\omega_1, \omega_2 \in \widehat{Q}_*$, so that D is a Banach algebra.

To prove that it is closed under the involution *, we use the manageability property of $\widehat{W}$. Namely, with P^{it} the scaling operator introduced in Theorem 3.8.(vii) of [30], we get for $\eta_1, \eta_2 \in \mathscr{H}$ and $\xi_1 \in \mathscr{D}(P^{-1/2}), \xi_2 \in \mathscr{D}(P^{1/2})$ that

$$\langle W^*(\xi_1 \otimes \eta_1), \xi_2 \otimes \eta_2 \rangle = \langle W(P^{-1/2}\xi_1 \otimes J_Q\eta_2), P^{1/2}\xi_2 \otimes J_Q\eta_1 \rangle.$$

Then if also $\eta_2 \in \mathscr{D}(P^{1/2})$ and $\eta_1 \in \mathscr{D}(P^{-1/2})$, we get by the commutation between W and $P^{it} \otimes P^{it}$, and between P^{it} and J_Q, that also

$$\langle W^*(\xi_1 \otimes \eta_1), \xi_2 \otimes \eta_2 \rangle = \langle W(\xi_1 \otimes J_Q P^{-1/2}\eta_2), \xi_2 \otimes J_Q P^{1/2}\eta_1 \rangle.$$

Hence $(\iota \otimes \omega_{\eta_2,\eta_1})(W)^* = (\iota \otimes \omega_{J_Q P^{-1/2}\eta_2, J_Q P^{1/2}\eta_1})(W)$, so that the closedness under involution follows from the fact that functionals of the form ω_{η_2,η_1} have dense span in Q_*.

□

By duality, the normclosure $\hat{D}$ of $\{(\omega \otimes \iota)(W) \mid \omega \in Q_*\}$ is a C*-algebra.

Proposition 11.2.2. *1. We have $d(L) \cup f(L) \subseteq M(D)$.*

2. W is a multiplier of $D \underset{min}{\otimes} \hat{D}$.

*3. Δ restricts to a (non-unital) *-homomorphism $D \to M(D \underset{min}{\otimes} D)$.*

4. The closure of the space $(D \otimes 1)\Delta(D)$ equals $(D \underset{min}{\otimes} D)\Delta(1)$, as does the closure of the space $(1 \otimes D)\Delta(D)$.

Proof. The first statement follows directly from the commutation relations in Lemma 11.1.2.

To prove the other assertions, we adapt the proof of the corresponding statement for manageable multiplicative unitaries as it appears in [105]. So to prove the second statement, we first show that it is enough to prove that $W \in M(D \underset{\text{min}}{\otimes} B_0(\mathscr{H}))$. For then of course also $\widehat{W} = \Sigma W^* \Sigma \in M(B_0(\mathscr{H}) \underset{\text{min}}{\otimes} \hat{D})$, which leads to

$$W_{12}^* W_{23} W_{12} W_{23}^* \in M(D \underset{\text{min}}{\otimes} B_0(\mathscr{H}) \underset{\text{min}}{\otimes} \hat{D}).$$

This implies

$$(\Delta(1) \otimes 1) W_{13} \in M(D \underset{\text{min}}{\otimes} B_0(\mathscr{H}) \underset{\text{min}}{\otimes} \hat{D})$$

by the third equality in Lemma 11.1.5. So

$$((\iota \otimes \omega)(\Delta(1)) \otimes 1) W \in M(D \underset{\text{min}}{\otimes} \hat{D})$$

for any $\omega \in B(\mathscr{H})_*$. But using the explicit form of $\Delta(1)$, it is easy to see that we can choose ω such that $(\iota \otimes \omega)(\Delta(1)) = 1$ (in fact $\epsilon \circ d^{-1}$ will do), and hence

$$W \in M(D \underset{\min}{\otimes} \widehat{D}).$$

So we prove now that $W \in M(D \underset{\min}{\otimes} B_0(\mathscr{H}))$. *We will write $W = V$, $D = D_V$ and $\mathscr{H} = \mathscr{G}$, since we will reuse part of this argument later on for a different value of V.* First note that the manageability of V needed in [105] (Definition 1.2) is given by the dual of the manageability formula in Theorem 3.8.(vii) of [30]):

$$\langle x \otimes u, V(z \otimes y)\rangle = \langle J_{\widehat{Q}} z \otimes P^{1/2} u, V^*(J_{\widehat{Q}} x \otimes P^{-1/2} y)\rangle$$

for $x, z \in \mathscr{G}, u \in \mathscr{D}(P^{1/2})$ and $y \in \mathscr{D}(P^{-1/2})$. Then one checks carefully that Proposition 4.1 of [105] is still valid. Further, as in Propositions 4.2 and 4.3 of [105], we can still conclude that

$$(1 \otimes \theta_z \theta_u^* \otimes \theta_x^*)(V_{12} W_{23}^*)(1 \otimes \theta_y \otimes 1) \in D_V \underset{\min}{\otimes} B_0(\mathscr{H}),$$

$$(1 \otimes \theta_z \theta_u^* \otimes \theta_x^*)(W_{23}^* V_{12})(1 \otimes \theta_y \otimes 1) \in D_V \underset{\min}{\otimes} B_0(\mathscr{H}),$$

where $u, x, y, z \in \mathscr{H}$ and where $\theta_x : \mathbb{C} \to \mathscr{H} : 1 \to x$, although we can *not* conclude the density statements in these propositions! Now note that by the first identity in Lemma 11.1.3 and by a pentagon equation for V, we have $(1 \otimes \Delta(1)) V_{12} W_{23}^* = W_{23}^* V_{12} V_{13}$, and so

$$\begin{aligned}&(1 \otimes \theta_z \theta_u^* \otimes \theta_x^*)((1 \otimes \Delta(1)) V_{12} W_{23}^*)(1 \otimes \theta_y \otimes 1)\\ = \;&(1 \otimes \theta_z \theta_u^* \otimes \theta_x^*)(W_{23}^* V_{12})(1 \otimes \theta_y \otimes 1) V\end{aligned}$$

for $x, y, u, z \in \mathscr{H}$.

Denote

$$K = [(1 \otimes \theta_z \theta_u^* \otimes \theta_x^*)(W_{23}^* V_{12})(1 \otimes \theta_y \otimes 1) \mid u, x, y, z \in \mathscr{H}],$$

where $[\,\cdot\,]$ denotes the normclosure of the linear span. Note that also

$$K = [((1 \otimes \theta_z)(1 \otimes \theta^*_{\hat{\Delta}^{\mathrm{op}}(1)(u \otimes x)})(V \otimes 1)(1 \otimes \theta_y \otimes 1)) \mid u, x, y, z \in \mathscr{H}],$$

and since $(1 \otimes \widehat{\Delta}^{\mathrm{op}}(1))V_{12} = V_{12}\widehat{\Delta}^{\mathrm{op}}(1)_{13}$ by the second identity of Lemma 11.1.3, we have

$$K = (D_V \underset{\min}{\otimes} B_0(\mathscr{H}))\widehat{\Delta}^{\mathrm{op}}(1).$$

It follows that

$$(D_V \underset{\min}{\otimes} B_0(\mathscr{H}))V \subseteq D_V \underset{\min}{\otimes} B_0(\mathscr{H}).$$

Now by Theorem 3.10.(v) and 3.11.(iii) of [30], we have that the modular conjugation $J_{\hat{Q}}$ for the dual left invariant weight implements the unitary antipode on Q, and that $(J_{\hat{Q}} \otimes J_Q)V(J_{\hat{Q}} \otimes J_Q) = V^*$. This implies that D is globally invariant under the unitary antipode. We then have

$$\begin{aligned}
(D \underset{\min}{\otimes} B_0(\mathscr{H}))V^* &= (D \underset{\min}{\otimes} B_0(\mathscr{H}))(J_{\hat{Q}} \otimes J_Q)V(J_{\hat{Q}} \otimes J_Q) \\
&= (J_{\hat{Q}} \otimes J_Q)(D \underset{\min}{\otimes} B_0(\mathscr{H}))V(J_{\hat{Q}} \otimes J_Q) \\
&\subseteq (J_{\hat{Q}} \otimes J_Q)(D \underset{\min}{\otimes} B_0(\mathscr{H}))(J_{\hat{Q}} \otimes J_Q) \\
&= D \underset{\min}{\otimes} B_0(\mathscr{H}),
\end{aligned}$$

which shows that $V \in M(D \underset{\min}{\otimes} B_0(\mathscr{H}))$. We then also get that

$$(D \underset{\min}{\otimes} B_0(\mathscr{H}))V = (D \underset{\min}{\otimes} B_0(\mathscr{H}))\Delta(1)$$

and

$$(D \underset{\min}{\otimes} B_0(\mathscr{H}))V^* = (D \underset{\min}{\otimes} B_0(\mathscr{H}))\widehat{\Delta}^{\mathrm{op}}(1).$$

We prove the third and fourth statement of the proposition together. Now denote

$$K = [(b \otimes 1)\Delta(a) \mid b, a \in D].$$

Then analogously as in Proposition 5.1 of [105], we get that

$$K = [(\iota \otimes \iota \otimes \omega)(x_{13}W_{13}W_{23}) \mid \omega \in B(\mathscr{H})_*, x \in D \otimes B_0(\mathscr{H})].$$

By the last result in the foregoing paragraph, we get that also

$$K = [(\iota \otimes \iota \otimes \omega)(x_{13}\Delta(1)_{13}W_{23}) \mid \omega \in B(\mathscr{H})_*, x \in D \underset{\min}{\otimes} B_0(\mathscr{H})].$$

As $\Delta(1)_{13}W_{23} = W_{23}(\Delta(1) \otimes 1)$ by the third identity of Lemma 11.1.3, we have

$$K = (D \underset{\min}{\otimes} D)\Delta(1).$$

By considering the opposite measured quantum groupoid, we also get that $(1 \otimes D)\Delta(D) = (D \underset{\min}{\otimes} D)\Delta(1)$.

□

Definition 11.2.3. *We call the couple (D, Δ) the weak Hopf C^*-algebra associated to the measured quantum groupoid $(L, Q, \Gamma, d, f, T, T', \epsilon)$.*

We will however say nothing more about its further structure (for example concerning invariant operator valued weights).

11.3 Universal weak Hopf C*-algebras

Now we look at an associated universal construction. For this section, we will follow closely the paper [54].

We still have at our disposition an adapted measured quantum groupoid $(L, Q, d, f, \Gamma, T, T', \epsilon)$ with L finite-dimensional and ϵ the canonical non-normalized Markov trace as in the first section. We keep writing $\mathscr{H}$ for $\mathscr{L}^2(Q)$. We will write

$$\mathscr{L}^1_*(Q) = \{\omega \in Q_* \mid \exists \theta \in Q_* : \forall x \in \mathscr{D}(S) : \theta(x) = \overline{\omega}(S(x))\},$$

where S denotes the antipode of $(L, Q, d, f, \Gamma, T, T', \epsilon)$ (defined in Theorem 3.8.(iv) of [30]), and where $\overline{\omega}(x) = \overline{\omega(x^*)}$ for $x \in Q, \omega \in Q_*$. When $\omega \in \mathscr{L}^1_*(Q)$, we will denote ω^* for the closure of $x \in \mathscr{D}(S) \to \overline{\omega}(S(x))$. Then $\mathscr{L}^1_*(Q)$ becomes a *-algebra if we also define multiplication as $\omega_1 \cdot \omega_2 = (\omega_1 \otimes \omega_2) \circ \Delta$ (which will be interior). Note that

$$\lambda : \mathscr{L}^1_*(Q) \to \hat{Q} : \omega \to (\omega \otimes \iota)(W)$$

is then a faithful non-degenerate *-representation of $\mathscr{L}^1_*(Q)$ (using $(\Delta \otimes \iota)(W) = W_{13}W_{23}$, and Theorem 3.8.(iv) of [30] for the fact that it is *-preserving). We can make $\mathscr{L}^1_*(Q)$ into a Banach *-algebra by the norm $\| \cdot \|_*$, where $\|\omega\|_* = \max\{\|\omega\|, \|\omega^*\|\}$ for $\omega \in \mathscr{L}^1_*(Q)$. Denote by $\hat{D}^u$ the universal C*-algebraic envelope of this *-algebra, and let $(\mathscr{H}^u, \lambda^u)$ be a faithful, non-degenerate representation for $\mathscr{L}^1_*(Q)$ such that the normclosure of $\lambda^u(\mathscr{L}^1_*(Q))$ may be identified with $\hat{D}^u$.

The first thing we want to do now, is construct a universal version W^u of W on $\mathscr{H} \otimes \mathscr{H}^u$, as is done in section 4 of [54]. Reading this section carefully, we see that the whole discussion goes through verbatim up to Lemma 4.3. We restate the main propositions.

First, some notation: if $\theta \in B(\mathscr{H})_*$, then, as in [54], we denote by $\lambda^*(\theta)$ the element $(\iota \otimes \theta)(W)$. For $\omega \in Q_*$ and $x \in Q$, we denote $\omega \cdot x = \omega(x\,\cdot)$ and $x \cdot \omega = \omega(\cdot\, x)$.

Proposition 11.3.1. *(Proposition 4.1 of [54]) There exists a unique *-representation μ of $\mathscr{L}^1_*(Q)$ on $\mathscr{H} \otimes \mathscr{H}^u$, such that*

$$\langle \mu(\omega)(\xi_1 \otimes \eta_1), \xi_2 \otimes \eta_2 \rangle = \langle \lambda^u(\omega \cdot \lambda^*(\omega_{\xi_1,\xi_2}))\eta_1, \eta_2 \rangle$$

for all $\omega \in \mathscr{L}^1_*(Q)$, $\xi_1, \xi_2 \in \mathscr{H}$ and $\eta_1, \eta_2 \in \mathscr{H}^u$.

Proof. As said, we can simply copy the proof in [54], because everything that is used, also holds in our setting (the operator V there being just our W). Note that we need the fact that W is a multiplier of $D \underset{\min}{\otimes} \widehat{D}$ for the convergence statement about the net $(\sum_{k \in M} x_{l'k} \otimes x_{kl})_{M \in F(K)}$ in that proof. □

Proposition 11.3.2. *(Corollary 4.1 of [54]) The set $\{\tilde{\omega} \in (\widehat{D}^u)^* \mid \exists y \in Q, \forall \omega \in \mathscr{L}^1_*(Q) : \tilde{\omega}(\lambda^u(\omega)) = \omega(y)\}$ is separating for $\widehat{D}^u$.*

Proof. As in [54]. The only thing to note maybe is that also in our setting,

$$\Delta \circ \tau_t = (\tau_t \otimes \tau_t) \circ \Delta$$

with τ_t the scaling group of $(L, Q, d, f, \Gamma, T, T', \epsilon)$, by Theorem 3.8.(ii) of [32]. □

Proposition 11.3.3. *(Lemma 4.3 in [54]) With s_λ the universal extension of λ to $\widehat{D}^u$ and s_μ the universal extension of μ to $\widehat{D}^u$, we have $\ker s_\lambda \subseteq \ker s_\mu$.*

Proof. First remark that the statement about the space

$$\mathcal{I} = \{\omega \in Q_* \mid \exists M \geqslant 0 \text{ with } |\omega(x^*)| \leqslant M \|\Lambda_\varphi(x)\| \quad \forall x \in \mathscr{N}_\varphi\}$$

just before the proof of Lemma 4.2 of [54], still holds true in our setting by Theorem 3.10.(v) of [30], noting that $Q^{d,f}_*$ equals Q_*. Now choose z in the Tomita algebra for $\hat{\varphi}$, where $\hat{\varphi}$ is the dual weight on $\widehat{Q}$ as defined in Theorem 3.10.(v) in [30]. Put $\xi = \Lambda_{\hat{\varphi}}(z)$ and choose $\eta \in \mathscr{H}$. Then for

$y \in \mathscr{N}_{\widehat{\varphi}}$, we have $|\omega_{\eta,\xi}(y^*)| = |\langle \eta, J_{\widehat{Q}}\sigma^{\widehat{\varphi}}_{i/2}(z)^* J_{\widehat{Q}}\Lambda_{\widehat{\varphi}}(y)\rangle|$, so that $\omega_{\eta,\xi} \in \widehat{\mathcal{I}}$ (defined as $\mathcal{I}$ but for the dual $\widehat{Q}$), and hence $(\omega_{\eta,\xi} \otimes \iota)(\widehat{W}) \in \mathscr{N}_{\varphi_Q}$ by Theorem 3.10.(v) and the biduality Theorem 3.11.(i) of [30]. So $\lambda^*(\omega_{\xi,\eta})^* \in \mathscr{N}_{\varphi_Q}$, since $\lambda^*(\omega_{\xi,\eta})^* = (\iota \otimes \omega_{\eta,\xi})(W^*) = (\omega_{\eta,\xi} \otimes \iota)(\widehat{W})$.

Then starting from the second paragraph, we can copy the proof of Lemma 4.3 of [54] (where unfortunately there are some *-signs missing), taking $\xi = v$ and $\eta = w$. Since also these ξ, η span a dense subspace of $\mathscr{H}$, we arrive at the final conclusion of the proof, because also in our situation $\mathcal{I} \cap \mathscr{L}^1_*(Q)$ is $\|\cdot\|_*$-norm dense in $\mathscr{L}^1_*(Q)$ (using for example standard smoothing arguments as in Lemma 4.2 of [54]).

□

The existence of a partial isometry $\mathcal{U} \in M(D \underset{\min}{\otimes} B_0(\mathscr{H} \otimes \mathscr{H}^u))$ as in Corollary 4.2 of [54] can also still be obtained, but we can no longer say the same things about its initial and final projection. The final Proposition 4.2 needs more reworking.

We proceed to fill up the gaps. Consider the maps

$$R^{\widehat{f}}_x : \mathscr{L}^1_*(Q) \to \mathscr{L}^1_*(Q) : \omega \to \omega \cdot f(x)$$

and

$$L^{\widehat{f}}_x : \mathscr{L}^1_*(Q) \to \mathscr{L}^1_*(Q) : \omega \to \omega \cdot d(x).$$

Then these are well-defined by Theorem 3.8.(i) and (ii) of [30], and for $\omega_1, \omega_2 \in \mathscr{L}^1_*(Q)$, $z \in Q$ and $x \in L$, we have

$$\begin{aligned}(\omega_1 \cdot (L^{\widehat{f}}_x(\omega_2)))(z) &= (\omega_1 \otimes \omega_2)((1 \otimes d(x))\Delta(z)) \\ &= (\omega_1 \otimes \omega_2)((f(x) \otimes 1)\Delta(z)) \\ &= ((R^{\widehat{f}}_x(\omega_1)) \cdot \omega_2)(z)\end{aligned}$$

by Lemma 11.1.4. Hence $m^{\widehat{f}}_x = (L^{\widehat{f}}_x, R^{\widehat{f}}_x)$ is a multiplier for $\mathscr{L}^1_*(Q)$.

It is easily seen that $m_{xy} = m_y \cdot m_x$. As for the *-structure, we have for $\omega \in \mathscr{L}^1_*(Q)$, $x \in L$ and $z \in \mathscr{D}(S)$ that

$$\begin{aligned}(\omega^* \cdot m^{\widehat{f}}_x)^*(z) &= \overline{\omega^*(f(x)S(z)^*)} \\ &= \omega(S(f(x))^* z) \\ &= \omega(d(x^*)z) \\ &= (m^{\widehat{f}}_{x^*} \cdot \omega)(z),\end{aligned}$$

using Theorem 3.8.(ii) of [30] in the third step, from which we can conclude that $L \to M(\mathscr{L}^1_*(Q)) : x \to m_x^{\hat{f}}$ is a unital anti-*-homomorphism (where $M(\mathscr{L}^1_*(Q))$ denotes the multiplier algebra for $\mathscr{L}^1_*(Q)$).

Suppose now that $(\tilde{\mathscr{H}}, \tilde{\pi}, \xi)$ is a cyclic *-representation of $\mathscr{L}^1_*(Q)$, and denote $\tilde{\omega} = \omega_{\xi,\xi} \circ \tilde{\pi}$. Choose $x \in L$ with $\|x\| \leqslant 1$. Then if $y \in L$ and $yy^* = 1 - xx^*$, we have for $\omega \in \mathscr{L}^1_*(Q)$ that

$$\begin{aligned}
\|\tilde{\pi}(m_x^{\hat{f}} \cdot \omega)\xi\|^2 &= \tilde{\omega}(\omega^*(m_x^{\hat{f}})^* m_x^{\hat{f}}\omega) \\
&= \tilde{\omega}(\omega^* m_{xx^*}^{\hat{f}}\omega) \\
&= \tilde{\omega}(\omega^*\omega) - \tilde{\omega}(\omega^*(m_y^{\hat{f}})^* m_y^{\hat{f}}\omega) \\
&\leqslant \|\tilde{\pi}(\omega)\xi\|,
\end{aligned}$$

and so we can anti-*-represent L on $\tilde{\mathscr{H}}$ by a map $\hat{b}$ such that for $x \in L$ and $z \in \mathscr{L}^1_*(Q)$, we have $\hat{b}(x)\tilde{\pi}(z) = \tilde{\pi}(m_x^{\hat{f}} \cdot z)$. This means that we can also anti-*-represent L on $\mathscr{H}^u$ by a map $\hat{f}^u$, such that $\hat{f}^u(x)\lambda^u(\omega) = \lambda^u(m_x^{\hat{f}} \cdot \omega)$ for all $x \in L$ and $\omega \in \mathscr{L}^1_*(Q)$. We should remark that this is compatible with the reduced case: we have $\hat{f}(x)\lambda(\omega) = \lambda(m_x^{\hat{f}} \cdot \omega)$ for $x \in L$, $\omega \in \mathscr{L}^1_*(Q)$, by the fourth identity in Definition 3.2 of [30].

In the same way, we can make for each $x \in L$ a multiplier $m_x^d = (L_x^d, R_x^d)$ on $\mathscr{L}^1_*(Q)$ by

$$L_x^d : \mathscr{L}^1_*(Q) \to \mathscr{L}^1_*(Q) : \omega \to d(x) \cdot \omega$$

and

$$R_x^d : \mathscr{L}^1_*(Q) \to \mathscr{L}^1_*(Q) : \omega \to f(x) \cdot \omega,$$

and we can make a universal *-representation d^u of L on $\mathscr{H}^u$ such that $d^u(x)\lambda^u(\omega) = \lambda^u(m_x^d \cdot \omega)$ for $\omega \in \mathscr{L}^1_*(Q)$.

We will now also denote $\tilde{\lambda}^u(d(x)) = d^u(x)$ and $\tilde{\lambda}^u(\hat{f}(x)) = \hat{f}^u(x)$.

Lemma 11.3.4. *For $\omega \in \mathscr{L}^1_*(Q)$, we have $(\iota \otimes \tilde{\lambda}^u)(\hat{\Delta}^{op}(1))\mu(\omega) = \mu(\omega)$.*

Proof. Choose $\omega \in \mathscr{L}^1_*(Q)$, $\xi_1, \xi_2 \in \mathscr{H}$, then

$$\begin{aligned}
&\sum_{i,j,l} n_l^{-1}\ \hat{f}(e^l_{ji})\lambda(\omega \cdot \lambda^*(\omega_{\xi_1, d(e^l_{ij})^*\xi_2})) \\
&\qquad = \sum_{i,j,l} n_l^{-1}\hat{f}(e^l_{ji})(\omega \otimes \iota)((\lambda^*(\omega_{\xi_1,d(e^l_{ij})^*\xi_2}) \otimes 1)W) \\
&\qquad = \sum_{i,j,l} n_l^{-1}\hat{f}(e^l_{ji})(\omega \otimes \iota \otimes \omega_{\xi_1,d(e^l_{ij})^*\xi_2})(W_{13}W_{12}) \\
&\qquad = (\omega \otimes \omega_{\xi_1,\xi_2} \otimes \iota)((1 \otimes \hat{\Delta}^{\mathrm{op}}(1))W_{12}W_{13}).
\end{aligned}$$

By the third identity in Lemma 11.1.3, and the fact that $\hat{\Delta}^{\mathrm{op}}(1)W = W$, we conclude

$$\sum_{i,j,l} n_l^{-1}\hat{f}(e^l_{ji})\lambda(\omega \cdot \lambda^*(\omega_{\xi_1,d(e^l_{ij})^*\xi_2})) = \lambda(\omega \cdot \lambda^*(\omega_{\xi_1,\xi_2})).$$

Applying $\lambda^u \circ \lambda^{-1}$, we find $(\iota \otimes \tilde{\lambda}^u)(\hat{\Delta}^{\mathrm{op}}(1))\mu(\omega) = \mu(\omega)$ by the defining property of μ. □

Proposition 11.3.5. *There exists a unique element $W^u \in M(D \underset{min}{\otimes} B_0(\mathscr{H}^u))$ such that $\lambda^u(\omega) = (\omega \otimes \iota)(W^u)$ for $\omega \in \mathscr{L}^1_*(Q)$.*

Proof. Define $\phi : \hat{D} \to B(\mathscr{H} \otimes \mathscr{H}^u)$ such that $\phi(s_\lambda(x)) = s_\mu(x)$ for all $x \in \hat{D}^u$, which is possible since $\ker s_\lambda \subseteq \ker s_\mu$. Then

$$U = (\iota \otimes \phi)(W) \in M(D \underset{\min}{\otimes} B_0(\mathscr{H} \otimes \mathscr{H}^u))$$

is well-defined (also writing ϕ for the extension to $M(\hat{D})$, which may be a non-unital map). Denote by p^u the projection of $\mathscr{H} \otimes \mathscr{H}^u$ onto the closure of $\mu(\mathscr{L}^1_*(Q))(\mathscr{H} \otimes \mathscr{H}^u)$, then $\phi(1) = p^u$.

Now fix $x \in L$ and $\omega \in \mathscr{L}^1_*(Q)$. Then we have $\phi((\omega \otimes \iota)(W)\hat{f}(x)) = \mu(\omega \cdot f(x))$, and for $\xi_1, \xi_2 \in \mathscr{H}$ and $\eta_1, \eta_2 \in \mathscr{H}^u$, we have

$$\langle \mu(\omega \cdot f(x))(\xi_1 \otimes \eta_1), \xi_2 \otimes \eta_2 \rangle = \langle \lambda^u(\omega \cdot (f(x)\lambda^*(\omega_{\xi_1,\xi_2})))\eta_1, \eta_2 \rangle.$$

But

$$\begin{aligned}
f(x)\lambda^*(\omega_{\xi_1,\xi_2}) &= f(x)((\iota \otimes \omega_{\xi_1,\xi_2})(W)) \\
&= (\iota \otimes \omega_{\xi_1,\xi_2})(W(1 \otimes \hat{f}(x))) \\
&= (\iota \otimes \omega_{\hat{f}(x)\xi_1,\xi_2})(W),
\end{aligned}$$

and so

$$\phi((\omega \otimes \iota)(W)\widehat{f}(x)) = \phi((\omega \otimes \iota)(W))(\widehat{f}(x) \otimes 1).$$

From this, we conclude

$$\phi(\widehat{f}(x)) = (\widehat{f}(x) \otimes 1)p^u.$$

On the other hand, $\phi(d(x)(\omega \otimes \iota)(W)) = \mu(d(x) \cdot \omega)$, and for $\xi_1, \xi_2 \in \mathscr{H}$ and $\eta_1, \eta_2 \in \mathscr{H}^u$, we have

$$\begin{aligned} \langle \mu(d(x) \cdot \omega)(\xi_1 \otimes \eta_1), \xi_2 \otimes \eta_2 \rangle &= \langle \lambda^u((d(x) \cdot \omega) \cdot \lambda^*(\omega_{\xi_1,\xi_2}))\eta_1, \eta_2 \rangle \\ &= \langle \lambda^u(d(x) \cdot (\omega \cdot \lambda^*(\omega_{\xi_1,\xi_2})))\eta_1, \eta_2 \rangle \\ &= \langle d^u(x)\lambda^u(\omega \cdot \lambda^*(\omega_{\xi_1,\xi_2}))\eta_1, \eta_2 \rangle, \end{aligned}$$

so that $\phi(d(x)) = (1 \otimes d^u(x))p^u$.

From this, it follows that

$$\begin{aligned} UU^* &= (\iota \otimes \phi)(\widehat{\Delta}^{\mathrm{op}}(1)) \\ &= \sum_{i,j,l} n_l^{-1} d(e^l_{ij}) \otimes \phi(\widehat{f}(e^l_{ji})) \\ &= \widehat{\Delta}^{\mathrm{op}}(1)_{12} p^u_{23}, \end{aligned}$$

and

$$\begin{aligned} U^*U &= (\iota \otimes \phi)(\Delta(1)) \\ &= \sum_{i,j,l} n_l^{-1} f(e^l_{ji}) \otimes \phi(d(e^l_{ij})) \\ &= ((\iota \otimes \tilde{\lambda}^u)\Delta(1))_{13} p^u_{23}. \end{aligned}$$

So if we consider $W^*_{12}U$, we see that it is still a partial isometry, since

$$\begin{aligned} U^* W_{12} W^*_{12} U &= U^* \widehat{\Delta}^{\mathrm{op}}_{12}(1) U \\ &= U^*U. \end{aligned}$$

We can choose $\omega' \in B(\mathscr{H})^+_*$ such that $(\omega' \otimes \iota)\widehat{\Delta}^{\mathrm{op}}(1) = 1$ (for example $\omega' = \epsilon \circ d^{-1}$), and then put

$$W^u = (\iota \otimes \omega' \otimes \iota)(W^*_{12}U).$$

If $\tilde{\omega} \in (\widehat{D}^u)^*$ is such that there exists $y \in Q$ with $\tilde{\omega}(\lambda^u(\omega)) = \omega(y)$ for all $\omega \in \mathscr{L}^1_*(Q)$, and $\rho \in B(\mathscr{H} \otimes \mathscr{H})_*$, we still have, as in Proposition 4.2 of [54], that $(\rho \otimes \iota)(U) \in \widehat{D}^u$ and $\tilde{\omega}((\rho \otimes \iota)(U)) = \rho(W(y \otimes 1))$. From this, we can conclude then that for $\omega_1, \omega_2 \in B(\mathscr{H})_*$, we have

$$\begin{aligned} \tilde{\omega}((\omega_1 \otimes \omega_2 \otimes \iota)(W_{12}^* U)) &= (\omega_1 \otimes \omega_2)(\Delta(1)(y \otimes 1)) \\ &= (\omega_1 \otimes \omega_2)(\sum_{i,j,l} n_l^{-1} f(e_{ji}^l) y \otimes d(e_{ij}^l)) \\ &= \sum_{i,j,l} n_l^{-1} \tilde{\omega}(\lambda^u(\omega_1 \cdot f(e_{ji}^l))) \omega_2(d(e_{ij}^l)) \\ &= \sum_{i,j,l} n_l^{-1} \tilde{\omega}(\lambda^u(\omega_1) \widehat{f}^u(e_{ji}^l)) \omega_2(d(e_{ij}^l)), \end{aligned}$$

from which we can deduce that

$$(\omega_1 \otimes \iota \otimes \iota)(W_{12}^* U) = (1 \otimes \lambda^u(\omega_1)) \cdot (\iota \otimes \tilde{\lambda}^u)(\widehat{\Delta}^{\mathrm{op}}(1)).$$

Applying $(\omega' \otimes \iota)$ to this last identity, we find that

$$(\omega_1 \otimes \iota)(W^u) = \lambda^u(\omega_1).$$

It is clear that $W^u \in M(D \underset{\min}{\otimes} B_0(\mathscr{H}^u))$ is uniquely determined by this property. ☐

Proposition 11.3.6. *The map W^u is a partial isometry with $(\iota \otimes \tilde{\lambda}^u)(\Delta(1))$ as its initial projection, and $(\iota \otimes \tilde{\lambda}^u)(\widehat{\Delta}^{op}(1))$ as its final projection.*

Proof. From $(\omega \otimes \iota \otimes \iota)(W_{12}^* U) = (1 \otimes \lambda^u(\omega)) \cdot (\iota \otimes \tilde{\lambda}^u)(\widehat{\Delta}^{\mathrm{op}}(1))$ and $(\omega \otimes \iota)(W^u) = \lambda^u(\omega)$ for all $\omega \in \mathscr{L}^1_*(Q)$, we deduce that

$$W_{12}^* U = W_{13}^u (1 \otimes (\iota \otimes \tilde{\lambda}^u)(\widehat{\Delta}^{\mathrm{op}}(1))).$$

For $x \in L$ and $\omega \in \mathscr{L}^1_*(Q)$, we have

$$\begin{aligned} (\omega \otimes \iota)(W^u) \widehat{f}^u(x) &= \lambda^u(\omega) \widehat{f}^u(x) \\ &= \lambda^u(\omega \cdot f(x)) \\ &= (\omega \otimes \iota)((f(x) \otimes 1) W^u), \end{aligned}$$

so $W^u(1 \otimes \widehat{f}^u(x)) = (f(x) \otimes 1)W^u$. This means that

$$W^u_{13}(1 \otimes (\iota \otimes \tilde{\lambda}^u)(\widehat{\Delta}^{\rm op}(1))) = (\Delta(1) \otimes 1)W^u_{13}.$$

So

$$\begin{aligned}(W^u_{13})^* W^u_{13}\ (1 \otimes (\iota \otimes \tilde{\lambda}^u)(\widehat{\Delta}^{\rm op}(1)))&\\ &= (1 \otimes (\iota \otimes \tilde{\lambda}^u)(\widehat{\Delta}^{\rm op}(1)))(W^u_{13})^* W^u_{13}(1 \otimes (\iota \otimes \tilde{\lambda}^u)(\widehat{\Delta}^{\rm op}(1)))\\ &= U^* W_{12} W^*_{12} U\\ &= ((\iota \otimes \tilde{\lambda}^u)(\Delta(1)))_{13} p^u_{23}.\end{aligned}$$

Applying $(\iota \otimes \omega' \otimes \iota)$ with ω' as before, we get as a first identity that

$$(W^u)^* W^u = ((\iota \otimes \tilde{\lambda}^u)(\Delta(1)))(1 \otimes (\omega' \otimes \iota)(p^u)).$$

On the other hand, by Lemma 11.3.4, $p^u \leqslant (\iota \otimes \tilde{\lambda}^u)(\widehat{\Delta}^{\rm op}(1))$. So also

$$(W^u_{13})^* W^u_{13} p^u_{23} = ((\iota \otimes \tilde{\lambda}^u)(\Delta(1)))_{13} p^u_{23},$$

and in particular, p^u_{23} commutes with $(W^u_{13})^* W^u_{13}$. Applying $(\iota \otimes \omega \otimes \iota)$ with $\omega \in B(\mathscr{H})_*$ arbitrary, we obtain

$$(W^u)^* W^u(1 \otimes (\omega \otimes \iota)(p^u)) = (\iota \otimes \tilde{\lambda}^u)(\Delta(1))(1 \otimes (\omega \otimes \iota)(p^u)).$$

Denote by $\tilde{p}^u$ the projection onto the closure of $\{(\omega \otimes \iota)(p^u)\mathscr{H}^u \mid \omega \in B(\mathscr{H})_*\}$. Then $p^u \leqslant (1 \otimes \tilde{p}^u)$, and $(1 \otimes 1 \otimes \tilde{p}^u)$ still commutes with $(W^u_{13})^* W^u_{13}$ and $((\iota \otimes \tilde{\lambda}^u)(\Delta(1)))_{13}$. Moreover, we get as a second identity that

$$(W^u)^* W^u(1 \otimes \tilde{p}^u) = ((\iota \otimes \tilde{\lambda}^u)(\Delta(1)))(1 \otimes \tilde{p}^u).$$

Putting the two identities together, we get

$$\begin{aligned}(W^u)^* W^u &= (W^u)^* W^u(1 \otimes \tilde{p}^u)\\ &= ((\iota \otimes \tilde{\lambda}^u)(\Delta(1)))(1 \otimes \tilde{p}^u).\end{aligned}$$

Now we repeat an argument of Proposition 4.2 of [54]: if $\tilde{p}^u$ were not equal to 1, we can find a non-zero $\eta \in \mathscr{H}^u$ such that $W^u(\xi \otimes \eta) = 0$ for all $\xi \in \mathscr{H}$. This implies that $(\omega \otimes \iota)(W^u)\eta = 0$ for all $\omega \in B(\mathscr{H})_*$. Since

$(\omega\otimes\iota)(W^u) = \lambda^u(\omega)$ for $\omega \in \mathscr{L}^1_*(Q)$, and λ^u is non-degenerate, we obtain a contradiction.

So we find that

$$(W^u)^*W^u = (\iota\otimes\tilde{\lambda}^u)(\Delta(1)).$$

Since

$$\begin{aligned}(\iota\otimes\iota\otimes\tilde{\lambda}^u)(\Delta(1)_{13}\widehat{\Delta}^{\mathrm{op}}(1)_{23}) &= (W^u_{13})^*W^u_{13}(1\otimes(\iota\otimes\tilde{\lambda}^u)(\widehat{\Delta}^{\mathrm{op}}(1))) \\ &= ((\iota\otimes\tilde{\lambda}^u)\Delta(1))_{13}p^u_{23},\end{aligned}$$

applying $(\omega''\otimes\iota\otimes\iota)$ for some $\omega'' \in Q_*$ with $(\omega''\otimes\iota)(\Delta(1)) = 1$ gives us that

$$p^u = (\iota\otimes\tilde{\lambda}^u)\widehat{\Delta}^{\mathrm{op}}(1).$$

Now we also get that

$$UU^* = \widehat{\Delta}^{\mathrm{op}}(1)_{12}(1\otimes(\iota\otimes\tilde{\lambda}^u)(\widehat{\Delta}^{\mathrm{op}}(1))),$$

and

$$\begin{aligned}W^*_{12}UU^*W_{12} &= W^*_{12}(1\otimes(\iota\otimes\tilde{\lambda}^u)(\widehat{\Delta}^{\mathrm{op}}(1)))W_{12} \\ &= (\Delta(1)\otimes 1)(\iota\otimes\iota\otimes\tilde{\lambda}^u)(\widehat{\Delta}^{\mathrm{op}}(1)_{13})\end{aligned}$$

by the second identity of Lemma 11.1.3. Then by the identities in the beginning of the proof,

$$\begin{aligned}(\Delta(1)\otimes 1)W^u_{13}(W^u_{13})^* &= W^u_{13}(1\otimes(\iota\otimes\tilde{\lambda}^u)(\widehat{\Delta}^{\mathrm{op}}(1)))(W^u_{13})^* \\ &= W^*_{12}UU^*W_{12} \\ &= (\Delta(1)\otimes 1)(\iota\otimes\iota\otimes\tilde{\lambda}^u)(\widehat{\Delta}^{\mathrm{op}}(1)_{13}),\end{aligned}$$

and applying $\iota\otimes\omega'''\otimes\iota$ for some $\omega''' \in B(\mathscr{H})_*$ with $(\iota\otimes\omega''')(\Delta(1)) = 1$ (for example $\epsilon\circ d^{-1}$ again), we get $W^u(W^u)^* = (\iota\otimes\tilde{\lambda}^u)(\widehat{\Delta}^{\mathrm{op}}(1))$, which finishes the proof.

□

We now give a different characterization for corepresentations of our special adapted measured quantum groupoids, by using partial isometries instead of unitaries. We will work with left corepresentations rather than the right corepresentations of section 5 of [30]. So let $\mathscr{G}$ be an L-L-bimodule by a *-representation a and anti-*-representation $\hat{b}$ of L.[3] Denote

$$q = \sum_{i,j,l} n_l^{-1} f(e^l_{ji}) \otimes a(e^l_{ij}),$$

$$q' = \sum_{i,j,l} n_l^{-1} d(e^l_{ij}) \otimes \hat{b}(e^l_{ji}).$$

Then exactly as in the first section of this chapter, $\mathscr{H} \,{}_{d}\underset{\epsilon^{\mathrm{op}}}{\otimes}{}_{\hat{b}} \mathscr{G}$ can be identified with $q'(\mathscr{H} \otimes \mathscr{G})$, and $\mathscr{H} \,{}_{f}\underset{\epsilon}{\otimes}{}_{a} \mathscr{G}$ with $q(\mathscr{H} \otimes \mathscr{G})$ (where we will now surpress the unitaries implementing the isomorphism). A unitary corepresentation

$$\tilde{V} : \mathscr{H} \,{}_{f}\underset{\epsilon}{\otimes}{}_{a} \mathscr{G} \rightarrow \mathscr{H} \,{}_{d}\underset{\epsilon^{\mathrm{op}}}{\otimes}{}_{\hat{b}} \mathscr{G}$$

in the sense of Definition 5.1 of [30] (adapted to the left setting) can thus be seen now as a partial isometry V in $Q \otimes B(\mathscr{G})$ with final projection q' and initial projection q. It satisfies, for all $x \in L$,

$$V(d(x) \otimes 1) = (1 \otimes a(x))V,$$

$$V(\hat{f}(x) \otimes 1) = (\hat{f}(x) \otimes 1)V$$

and

$$V(1 \otimes \hat{b}(x)) = (f(x) \otimes 1)V,$$

as well as the identity

$$(\Delta \otimes \iota)V = V_{13}V_{23}.$$

Conversely, any such partial isometry in $Q \otimes B(\mathscr{G})$ satisfying these four relations, and having q' and q as resp. final and initial projection, determines a unitary corepresentation $\tilde{V}$ by restriction.

It is easy to see that the partial isometry W^u satisfies these conditions: on $\mathscr{H}^u$, we take the L-L-bimodule structure given by d^u and $\hat{f}^u$ (which are easily seen to commute by definition). Then the initial and final projections of W^u satisfy the right conditions for a corepresentation, by Proposition

[3] We do not assume that the L-L-bimodule is faithful, unlike in [30]. This causes no problems however.

11.3.6. Further, we also know that $W^u \in Q \otimes B(\mathscr{H}^u)$, and since for $\omega_1, \omega_2 \in \mathscr{L}^1_*(Q)$, we have

$$\begin{aligned}((\omega_1 \otimes \omega_2) \circ \Delta) \otimes \iota)(W^u) &= \lambda^u(\omega_1 \cdot \omega_2) \\ &= \lambda^u(\omega_1)\lambda^u(\omega_2) \\ &= (\omega_1 \otimes \iota)(W^u)(\omega_2 \otimes \iota)(W^u),\end{aligned}$$

we can also conclude that

$$(\Delta \otimes \iota)(W^u) = W^u_{13} W^u_{23}.$$

To end, we have shown in the proof of Proposition 11.3.6 that for $x \in L$, we have

$$W^u(1 \otimes \hat{f}^u(x)) = (f(x) \otimes 1)W^u,$$

and

$$W^u(d(x) \otimes 1) = (1 \otimes d^u(x))W^u$$

follows by a similar argument. The final commutation needed, with $\hat{f}(x)$ on the left, can be deduced for example by the following argument: using notation as in the proof of Proposition 11.3.6, it is enough to show that $\hat{f}(x) \otimes 1 \otimes 1$ commutes with $W^*_{12}U$, since $W^*_{12}U = W^u_{13}(1 \otimes (\iota \otimes \tilde{\lambda}^u)(\hat{\Delta}^{\mathrm{op}}(1)))$. But since $U = (\iota \otimes \phi)(W)$, this follows at once from the fact that $(\hat{f}(x) \otimes 1)$ commutes with W.

There is only one thing we still have to do, before we can draw our final conclusion. Namely, we have to show that W^u lives in its expected C*-algebraic home.

Proposition 11.3.7. *We have* $W^u \in M(D \underset{min}{\otimes} \hat{D}^u)$.

Proof. In fact, take *any* partially isometric (for convenience sake) right corepresentation $V \in B(\mathscr{G} \otimes \mathscr{H})$ of $(L, Q, d, f, \Gamma, T, T', \epsilon)$, and denote by D_V the normclosure of its first leg: $D_V = [(\iota \otimes \omega)(V) \mid \omega \in Q_*]$. (In case $V = \Sigma W^u \Sigma$, which is a right corepresentation for the *opposite* quantum groupoid, it is clear that $D_V = \hat{D}^u$ by Proposition 11.3.5.) Then D_V evidently becomes a Banach algebra by the corepresentation property of V. It will be a C*-algebra even, by the manageability of V (Theorem 5.11 of [30]):

$$\langle V(s \otimes y), r \otimes u \rangle = \langle V^*(s \otimes J_{\hat{Q}} P^{-1/2} u), r \otimes J_{\hat{Q}} P^{1/2} y \rangle$$

for all $r, s \in \mathscr{H}^u, y \in \mathscr{D}(P^{1/2})$ and $u \in \mathscr{D}(P^{-1/2})$, so that

$$(\iota \otimes \omega_{J_{\hat{Q}} P^{-1/2} u, J_{\hat{Q}} P^{1/2} y})(V)^* = (\iota \otimes \omega_{y,u})(V).$$

We want to show that $V \in M(D_V \underset{\min}{\otimes} D)$. It is then again enough to show that $V \in M(D_V \underset{\min}{\otimes} B_0(\mathscr{H}))$, since

$$V_{12}^* W_{23}^* V_{12} W_{23} = (\widehat{\Delta}^{\mathrm{op}}(1) \otimes 1) V_{13},$$

with $W \in M(B_0(\mathscr{H}) \underset{\min}{\otimes} D)$ (where for notational convenience we have dropped the representation symbols for the left and right representation of L on $\mathscr{G}$ associated to V).

Now the corresponding part of Proposition 11.2.2 applies word for word, up to the point where we have shown that $(D_V \underset{\min}{\otimes} B_0(\mathscr{H}))V \subseteq D_V \underset{\min}{\otimes} B_0(\mathscr{H})$. But since V^* is a right corepresentation for the *opposite* measured quantum groupoid (cf. Theorem 3.12.(i) of [30]; this opposite measured quantum groupoid is still of our special form), and since $D_V = D_{V^*}$, we also have $(D_V \underset{\min}{\otimes} B_0(\mathscr{H}))V^* \subseteq D_V \underset{\min}{\otimes} B_0(\mathscr{H})$. This concludes the proof. □

Now we can state the main result:

Proposition 11.3.8. *There is a one-to-one-correspondence between left corepresentations of* (L, Q, d, f, Δ) *and non-degenerate* **-representations of* $\widehat{D}^u$.

Proof. If V is a left corepresentation for $(L, Q, d, f, \Gamma, T, T', \epsilon)$ (given in the form of a partial isometry), it is clear by Propositions 5.5 and 5.10 of [30] that $\omega \to (\omega \otimes \iota)(V)$ determines a non-degenerate *-representation of $\mathscr{L}^1_*(Q)$.

Conversely, let $\tilde{\pi}$ be a non-degenerate *-representation of $\widehat{D}^u$. As we have seen, this comes equipped with a *-representation a and anti-*representation $\widehat{b}$ of L on $\mathscr{G}$. Let $V = (\iota \otimes \tilde{\pi})(W^u)$, which is a well-defined partial isometry in $M(D \underset{\min}{\otimes} B_0(\mathscr{G}))$ by the previous results. Moreover, as necessarily $a(x) = \tilde{\pi}(d^u(x))$ and $\widehat{b}(x) = \tilde{\pi}(\widehat{f}^u(x))$ for $x \in L$ by the non-degeneracy of the representations, we see that V satisfies the right properties with respect to its initial and final projection, and that it satisfies the right commutation relations with respect to a and $\widehat{b}$. Since $(\Delta \otimes \iota)(V) = V_{13} V_{23}$, we get that V is indeed a left co-representation.

Of course, both operations are also inverses of each other.

□

Nederlandse samenvatting

Het kernbegrip in deze thesis is de 'comonoïdale Morita equivalentie'. We belichten dit concept vanuit drie standpunten. Vooreerst voeren we dit begrip in voor Hopf algebra's. Daarna bestuderen we het voor de algebraïsche en *-algebraïsche kwantumgroepen van Van Daele ([93]). Dit beslaat het eerste deel van onze thesis, dat enkel gebruik maakt van (elementaire) algebraïsche technieken. In het tweede deel bestuderen we dan comonoïdale Morita theorie voor de lokaal compacte kwantumgroepen van Kustermans en Vaes ([56]), en gebruiken hiertoe extensief de theorie van von Neumann algebra's en gewichten (niet-commutatieve integratietheorie).

We geven nu wat meer uitleg over deze begrippen, en over de resultaten die in deze thesis behaald werden.

De volgende symbolen zullen vaak gebruikt worden. Met k wordt een (willekeurig) veld bedoeld, en met $\odot$ het tensorproduct over k. Met ι wordt altijd het 'identiteitsmorfisme' aangeduid. Als $S \subseteq B(\mathscr{H})$ een verzameling van begrensde operatoren op een zekere Hilbertruimte $\mathscr{H}$ is, dan duiden we met S' zijn *commutant* aan, i.e. de verzameling van alle begrensde operatoren op $\mathscr{H}$ die met elk element uit S commuteren. Met $\otimes$ duiden we het tensorproduct van Hilbertruimtes en het (spatiaal) tensorproduct van von Neumann algebra's aan. Met Σ noteren we de 'volta': als $\mathscr{H}$ en $\mathscr{G}$ bijvoorbeeld twee Hilbertruimtes zijn, dan is $\Sigma : \mathscr{H} \otimes \mathscr{G} \to \mathscr{G} \otimes \mathscr{H}$ de afbeelding die $\xi \otimes \eta$ afstuurt op $\eta \otimes \xi$.

N.1 Morita theorie voor Hopf algebra's

Het eerste hoofdstuk van deze thesis is bedoeld als inleiding, motivatie en intuïtie met betrekking tot de theorie die in latere hoofdstukken ontwikkeld wordt. We beginnen met het invoeren van de welbekende notie van Morita equivalentie tussen (unitale associatieve) algebra's A en D (over een veld k). We presenteren drie equivalente definities: één categorisch gekleurde, één concrete maar asymmetrische, en één concrete *en* symmetrische definitie. Dit zijn met name, respectievelijk, 'het bestaan van een (k-lineaire) equivalentie tussen de module-categorieën van A en D', 'het bestaan van een getrouw projectieve, eindig voortgebrachte rechtse A-module met (een kopie van) D als endomorfisme-groep', en tenslotte 'het bestaan van een link algebra tussen A en D'.

De eerste van deze definities is de originele, zoals ingevoerd door Morita in de jaren '50. Het is deze karakterisatie die de 'betekenis' van Morita equivalentie laat zien: als men een abelse (k-lineaire) categorie beschouwt als een kwantisatie van een klassiek schema over k (in de algebro-geometrische betekenis), dan kan het best zijn dat, indien de abelse categorie 'affien' is, er *meerdere*, niet-isomorfe k-algebra's zijn die deze categorie als spectrum (i.e. als module-categorie) hebben. Zo heeft een gewoon *punt* reeds een hele rij van niet-commutatieve realisaties (=representaties), namelijk de n-bij-n-matrices over k. Men kan dan bijvoorbeeld een eigenschap van een algebra een eigenschap van de onderliggende kwantumruimte noemen, als ze stabiel is onder Morita equivalentie. (Merk op dat dit slechts één interpretatie is: men kan evengoed de algebra's *zelf* als (functie-algebra's van) kwantumruimtes zien, en de bijhorende module-categorie als een grote, maar incomplete invariant.)

De tweede definitie van Morita equivalentie legt minder nadruk op het equivalentie-aspect (het is bijvoorbeeld niet eens zonder meer duidelijk uit deze definitie of Morita equivalentie werkelijk een equivalentie-relatie tussen algebra's bepaalt). Ze geeft eerder een handige constructiemethode om, gegeven een algebra, een Morita equivalente algebra te *creëren*. Inderdaad: gegeven een algebra en een getrouw projectieve, eindig voortgebrachte rechtse A-module B, geeft $D = \mathrm{End}_A(B_A)$ meteen een met A Morita equivalente algebra. Het is precies dit reconstructie-aspect dat verderop in de thesis in meer complexere situaties behandeld wordt, en het meeste technische werk vergt.

De derde definitie tenslotte is vooral interessant ten aanzien van veralgemeningen. Met een link algebra tussen twee algebra's wordt een unitale algebra E met vaste projectie (= idempotent element) $e \in E$ bedoeld, zodat zowel e als zijn complement $1_E - e$ *vol* zijn (i.e. er bestaat geen 2-zijdig ideaal dat één van deze projecties bevat), en zodat de 'hoeken' van E, i.e. de algebra's eEe en $(1_E - e)E(1_E - e)$, isomorf zijn, als algebra, met respectievelijk A en D. Het voordeel van deze definitie, naast het feit dat ze volledig symmetrisch is ten opzichte van A en D, is dat er enkel gebruik wordt gemaakt van algebra's. Zoals gezegd leidt dit op een erg eenvoudige wijze tot goede veralgemeningen, zoals bijvoorbeeld een Morita equivalentie tussen niet-unitale algebra's van een bepaald type: men eist dan dat er een link algebra tussen hen bestaat van hetzelfde type. Deze notie werd bij mijn weten vooral gebruikt in de operator-algebraïsche context (zie [67] waar het begrip ingevoerd wordt in de C*-algebraïsche context). Merk op dat een link algebra E zelf *ook* Morita equivalent is met beide algebra's A en D waartussen het een Morita equivalentie creëert. We komen hier later nog even op terug.

Nu gaan we wat meer structuur plaatsen op de algebra's die Morita equivalent zijn: we voorzien ze van een Hopf algebra structuur (met bijectieve antipode).

Definitie N.1.1. *Een koppel* (A, Δ_A) *wordt een* Hopf algebra *genoemd, als* A *een unitale algebra is, voorzien van een unitaal homomorfisme* $\Delta_A : A \to A \odot A$*, de* covermenigvuldiging*, dat voldoet aan de volgende* coassociativiteitsvoorwaarde*:*

$$(\Delta_A \otimes \iota_A)\Delta_A = (\iota_A \otimes \Delta_A)\Delta_A.$$

Verder moet er een unitaal homomorfisme $\varepsilon_A : A \to k$ *bestaan, de* coeenheid*, en een bijectieve lineaire afbeelding* $S_A : A \to A$*, de* antipode*, zodat*

$$(\varepsilon_A \otimes \iota_A)\Delta_A = \iota_A = (\iota_A \otimes \varepsilon_A)\Delta_A$$

en

$$S_A(a_{(1)})a_{(2)} = \varepsilon(a)1_A = a_{(1)}S_A(a_{(2)}).$$

We hebben hierbij ondertussen de Sweedler-notatie ingevoerd: men schrijft dan (formeel) $\Delta_A(a) = a_{(1)} \otimes a_{(2)}$, wat toelaat om berekeningen met de covermenigvuldiging sterk te vereenvoudigen. Hopf algebra's kunnen gezien worden als niet-commutatieve veralgemeningen van de (polynomiale) functiealgebra's op (algebraïsche) groepen, waarbij bijvoorbeeld de covermenigvuldiging nu de rol van de groepsvermenigvuldiging speelt. We merken op dat S_A anti-multiplicatief is ($S_A(aa') = S_A(a')S_A(a)$), en ook anti-comultiplicatief:

$$\Delta_A(S_A(a)) = S_A(a_{(2)}) \otimes S_A(a_{(1)}).$$

Men kan nu op zoek gaan naar een notie van Morita equivalentie tussen *Hopf* algebra's die de extra structuur in het oog houdt. We zullen deze equivalentie 'comonoïdale Morita equivalentie' noemen (al merken we op dat er in de literatuur reeds andere terminologieën voorhanden zijn). Ze kan opnieuw op drie manieren gekarakteriseerd worden, net als gewone Morita equivalentie.

Vooreerst is er de meest natuurlijke, categorische definitie. Voor Hopf algebra's verkrijgt de categorie van modules voor de onderliggende algebra namelijk een extra structuur: het wordt een *monoïdale categorie*. Dit betekent dat de categorie voorzien is van een bifunctor $\otimes$, die ('op compatibele isomorfismes na') associatief is. In ons geval is deze bifunctor het gewone tensorproduct van vectorruimtes, waarop een module-structuur gecreëerd wordt met behulp van de covermenigvuldiging: als A de Hopf algebra is en V en W twee (linkse) modules, dan definieert men de volgende A-module structuur op $V \odot W$:

$$a \cdot (v \otimes w) := \Delta_A(a) \cdot (v \otimes w),$$

waarbij we in het rechterlid op het linkse been van $\Delta_A(a)$ de V-module structuur toepassen, en op het rechtse been de W-module structuur.

Er is nu een natuurlijke notie van comonoïdale equivalentie tussen monoïdale categorieën $(\mathcal{C}, \otimes)$ en $(\mathcal{D}, \otimes)$ (waarbij we voor de notationele eenvoud de tensorproducten niet verder labelen): men eist dat er een equivalentie F tussen beide categorieën bestaat, zodat de functoren $F \circ \otimes$ en $\otimes \circ (F \times F)$ van het Cartesisch product $\mathcal{C} \times \mathcal{C}$ naar $\mathcal{D}$ natuurlijk isomorf zijn via een natuurlijk isomorfisme u dat verder aan de volgende vergelijking voldoet (die de *2-cocykel identiteit* wordt genoemd):

$$(u_{X,Y} \otimes \iota_{F(Z)})u_{X\otimes Y,Z} = (\iota_{F(X)} \otimes u_{Y,Z})u_{X,Y\otimes Z}.$$

We zeggen nu dat twee Hopf algebra's *comonoïdaal Morita equivalent zijn* als hun module-categorieën comonoïdaal equivalent zijn.

De tweede definitie voor comonoïdale Morita equivalentie is opnieuw asymmetrisch van aard. We voeren hiertoe de volgende notie in.

Definitie N.1.2. *Zij A een Hopf algebra. Een* comonoïdale (rechtse) Morita module *voor A is een rechtse A-module B, voorzien van een coassociatieve lineaire afbeelding $\Delta_B : B \to B \odot B$, zodat*

$$\Delta_B(ba) = \Delta_B(b)\Delta_A(a), \qquad \text{voor alle } a \in A, b \in B,$$

en zodat de afbeelding

$$B \odot A \to B \odot B : b \otimes a \to \Delta_B(b)(1 \otimes a)$$

een (lineair) isomorfisme is.

We zeggen dan dat twee Hopf algebra's A en D comonoïdaal Morita equivalent zijn, als er een comonoïdale rechtse Morita A-module bestaat, zó dat $D \cong \mathrm{End}_A(B_A)$, en zó dat, als we D identificeren met zijn beeld onder dit isomorfisme, $\Delta_B(d \cdot b) = \Delta_D(d) \cdot \Delta_B(b)$. *Automatisch* volgt hieruit dat B dan getrouw projectief en eindig voortgebracht is over A, zodat we in ieder geval reeds weten dat D en A als algebra's Morita equivalent zijn. We noemen zo een B dan een *comonoïdale equivalentie bimodule* tussen de Hopf algebra's A en D. We zien dat deze definitie opnieuw eerder gericht is op de *constructie* van comonoïdale equivalenties: we tonen in de thesis inderdaad aan dat een comonoïdale Morita module gecompleteerd kan worden tot een comonoïdale equivalentie bimodule. In het bijzonder kan dus vanuit de A-module B een Hopf algebra D *gemaakt* worden.

De derde definitie van comonoïdale Morita equivalentie houdt in, dat de twee Hopf algebra's ingebed moeten zijn als hoeken van een zekere *zwakke* Hopf algebra E, die we de *zwakke Hopf link algebra* noemen. Een zwakke Hopf algebra is een veralgemening van het begrip Hopf algebra, waarbij bijvoorbeeld niet langer geëist wordt dat de covermenigvuldiging eenheidbewarend is (zie [11]). Zwakke Hopf algebra's kunnen gezien worden als de niet-commutatieve versies van 'affiene groepoïde-schema's met een eindige set objecten'. In het geval van comonoïdale Morita equivalentie kan de zwakke Hopf link algebra als volgt geïnterpreerd worden: het is een kwantumgroepoïde met twee klassieke objecten, zodat de twee comonoïdaal equivalente Hopf algebra's de rol spelen van groepalgebra's van de endomorfismegroepen van de twee objecten, en zodat de anti-diagonale hoeken van E (i.e. $(1_E - e)Ee$ en $eE(1_E - e)$) de rol spelen van 'pijl-bimodules' voor de morfismes *tussen* de twee objecten. Formeel vertaalt dit zich onder andere in het feit dat de onderliggende algebra voor de zwakke Hopf link algebra de structuur van een link algebra heeft, op zodanige wijze dat de bijhorende projectie e én zijn complement beide groepsgelijkende elementen zijn (dus bijvoorbeeld $\Delta_E(e) = e \otimes e$).

Naast de notie van 'comonoïdale Morita equivalentie' is er ook de 'monoïdale co-Morita equivalentie' tussen Hopf algebra's. *Formeel* is deze theorie volkomen duaal aan de vorige, en in het geval van *eindig dimensionale* Hopf algebra's is deze dualiteit bijvoorbeeld zelfs *meer* dan louter formeel: men schakelt van de ene theorie naar de andere over door duales van vectorruimtes te nemen, en alle structuur via transpositie over te brengen. Opnieuw is er een drievuldigheid aan definities voor monoïdale co-Morita equivalentie beschikbaar, die we nu niet meer allemaal in extenso zullen bespreken: categorisch zal dit neerkomen op het monoïdaal equivalent zijn van de *co*-module categorieën van de beide Hopf algebra's, terwijl concreet men het bestaan van een 'bi-Galois object' of 'zwakke Hopf co-link algebra' tussen de twee Hopf algebra's eist. We gaan enkel de theorie van (bi-)Galois objecten nog wat nader toelichten.

Een (rechts) Galois object voor een Hopf algebra is formeel duaal aan een comonoïdale Morita module. 'Transponeren' we de structuur van deze laatste, dan bekomen we de volgende definitie.

Definitie N.1.3. *Zij A een Hopf algebra. Een* (rechts) Galois object *voor A is een unitale algebra B, voorzien van een rechtse coactie α_B, i.e. een unitaal homomorfisme $\alpha_B : B \to B \odot A$ dat voldoet aan*

$$(\iota \otimes \Delta_A)\alpha_B = (\alpha_B \otimes \iota_A)\alpha_B \qquad \textit{(coactie eigenschap)},$$

zó dat de afbeelding

$$B \odot B \to B \odot A : b \otimes b' \to (b \otimes 1)\alpha_B(b')$$

een bijectie is.

Als A en D twee Hopf algebra's zijn, dan is een bi-Galois object *tussen A en D een unitale algebra B voorzien van een rechste A-Galois object structuur α_B en een linkse D-Galois object structuur $\gamma_B : B \to D \odot B$, zó dat α_B en γ_B* commuteren*:*

$$(\gamma_B \otimes \iota_A)\alpha_B = (\iota_D \otimes \alpha_B)\gamma_B.$$

De theorie van (bi-)Galois objecten werd uitvoerig behandeld in het artikel [71]. Één van de belangrijke stellingen in dat artikel betreft opnieuw een reconstructie-resultaat: een rechts Galois object kan uniek gecompleteerd worden tot een bi-Galois object. In het bijzonder kan dus ook uit een Galois object een 'nieuwe' Hopf algebra geconstrueerd worden (die natuurlijk isomorf zou *kunnen* zijn met de oorspronkelijke Hopf algebra). We willen ook vermelden dat er een meetkundige intepretatie voor Galois objecten is: ze kunnen gezien worden als niet-commutatieve versies van hoofdvezelbundels ('principal fiber bundles') *over een punt*. Dit merkwaardig, omdat dit concept in de puur klassieke context van een bedrieglijke eenvoud is: als we ter voorbeeld ons beperken tot *eindige* groepen, dan is een hoofdvezelbundel over een punt, met een eindige groep $\mathfrak{G}$ als structuurgroep, niets anders dan een eindige verzameling X, voorzien van een (rechtse) actie van $\mathfrak{G}$, zó dat deze actie zowel transitief (er is slechts één orbiet) als vrij ($\mathfrak{G}$ ageert trouw op elke orbiet) is. Met andere woorden, X '*is*' gewoon de groep $\mathfrak{G}$ zelf, voorzien van de actie via rechtse translatie. Er is echter de volgende subtiliteit: het isomorfisme tussen X en $\mathfrak{G}$ is *niet natuurlijk*: men moet één van de punten van X het label 'eenheid' toekennen! Dit feit kan gezien worden als een zwakke weerspiegeling van het vreemde gedrag dat mogelijk is in de kwantum-context. (We moeten hierbij natuurlijk opmerken dat er wel degelijk bi-Galois objecten bestaan tussen bepaalde niet-isomorfe Hopf algebra's. Een mooie klasse van voorbeelden werd geconstrueerd in [9] (zie ook [10] voor voorbeelden in een meer operator-algebraïsch kader).).

N.2 Galois objecten voor algebraïsche kwantumgroepen

In de volgende drie hoofdstukken van de thesis (hoofdstukken 2 tot en met 4) ontwikkelen we (in essentie) een theorie van (co-)monoïdale (co-)Morita equivalentie voor algebraïsche kwantumgroepen.

In het tweede hoofdstuk van onze thesis worden de belangrijkste definities en resultaten uit [92] en [93] uiteengezet. In het artikel [93] wordt de notie van 'algebraïsche kwantumgroep' ingevoerd. Dit is een object dat tegelijkertijd een veralgemening als een specialisatie van een Hopf algebra is. Namelijk: het is een veralgemening omdat er niet langer geëist wordt dat de onderliggende algebra een eenheid heeft, maar het is ook een specialisatie omdat men het bestaan van een *invariante functionaal* aanneemt. Men moet deze functionaal zien als het analogon van een (linkse) Haarmaat op een gewone lokaal compacte groep.

Om de definitie van een algebraïsche kwantumgroep te kunnen formuleren, moeten we enkele begrippen aangaande niet-unitale algebra's invoeren.

Definitie N.2.1. *Zij A een (associatieve) algebra, mogelijk zonder eenheid.*

We noemen A niet-ontaard *als A een trouwe linkse* en *rechtse module over zichzelf is. Met andere woorden, als $a \in A$ voldoet aan $aa' = 0$ voor alle $a' \in A$, dan is $a = 0$, en evenzo als $a'a = 0$ voor alle $a' \in A$.*

We noemen A idempotent *als $A \cdot A = A$.*

Definitie N.2.2. *Zij A een algebra, mogelijk zonder eenheid. De* algebra $M(A)$ van vermenigvuldigers voor *A ('multiplier algebra') bestaat uit koppels $m = (l_m, r_m)$, met l_m, r_m lineaire afbeeldingen $A \to A$ die voldoen aan*

$$a' l_m(a) = r_m(a')a$$

voor alle $a, a' \in A$. We noteren dan $l_m(a) = m \cdot a$ en $r_m(a) = a \cdot m$, zodat bovenstaande gelijkheid een associativiteits-eigenschap uitdrukt:

$$(a' \cdot m) \cdot a = a' \cdot (m \cdot a).$$

Merk op dat als A een niet-ontaarde algebra is, we A kunnen vereenzelvigen met een deel van $M(A)$, door a af te beelden op (l_a, r_a), waarbij l_a de operatie links en r_a de operatie rechts vermenigvuldigen met a voorstelt.

De volgende definitie geeft aan wat de goede notie van morfismes tussen niet-ontaarde algebra's is.

Definitie N.2.3. *Zij A en B niet-ontaarde algebra's, en $f : A \to M(B)$ een homomorfisme.*

We zeggen dat f de unieke unitale extensie-eigenschap heeft *of* u.u.e. *is, als $f(A)B = B = Bf(A)$.*

We zeggen dat f de unieke extensie-eigenschap heeft *of u.e. is, als er een idempotent $p \in M(B)$ bestaat zodat $f(A)B = pB$ en $Bf(A) = Bp$.*

De belangrijkste eigenschap van een u.u.e. homomorfisme is dat ze inderdaad een unieke uitbreiding tot een unitaal homomorfisme $M(A) \to M(B)$ heeft (terwijl een u.e. homomorfisme uitbreidbaar is tot een homomorfisme $M(A) \to M(B)$ die $1_{M(A)}$ afstuurt op de idempotent p waarvan sprake in de definitie). Verder merken we op dat als A een niet-ontaarde idempotente algebra is, de identiteitsafbeelding voor A u.u.e. is, en dat het tensorproduct van (u.)u.e. afbeeldingen opnieuw (u.)u.e. is.

We kunnen nu de definitie van een algebraïsche kwantumgroep geven.

Definitie N.2.4. *Zij A een niet-ontaarde, idempotente algebra, voorzien van een u.u.e. homomorfisme $\Delta_A : A \to M(A \odot A)$. We noemen (A, Δ_A) een* algebraïsche kwantumgroep *als Δ_A aan de coassociativiteits-voorwaarde $(\Delta_A \otimes \iota_A)\Delta_A = (\iota_A \otimes \Delta_A)\Delta_A$ voldoet*[4]*, als de afbeeldingen*

$$T_{\Delta_A,2} : A \odot A \to A \odot A : a \otimes a' \to \Delta_A(a)(1 \otimes a'),$$

$$T_{1,\Delta_A} : A \odot A \to A \odot A : a \otimes a' \to (a \otimes 1)\Delta_A(a'),$$

$$T_{\Delta_A,1} : A \odot A \to A \odot A : a \otimes a' \to \Delta_A(a)(a' \otimes 1),$$

$$T_{2,\Delta_A} : A \odot A \to A \odot A : a \otimes a' \to (1 \otimes a)\Delta_A(a')$$

allen bijectief zijn[5]*, en als er een niet-triviale functionaal $\varphi_A : A \to k$ bestaat die voldoet aan*

$$(\iota \otimes \varphi_A)(\Delta_A(a)) = \varphi_A(a)1_A$$

voor alle $a \in A$, waarbij het linkerlid op natuurlijke wijze geïnterpreteerd kan worden als een vermenigvuldiger voor a.

In [93] wordt dan aangetoond dat deze algebraïsche kwantumgroepen een verrassend rijke structuur hebben. *Vooreerst* is er de verdere structuur van gewone Hopf algebra's aanwezig, met name een co-eenheid ε_A en inverteerbare antipode $S_A : A \to A$. *Ten tweede* blijkt de links invariante functionaal φ_A automatisch uniek te zijn (op vermenigvuldigen met een niet-nul element uit k na), en te voldoen aan de volgende twee sterke eigenschappen: φ_A is *getrouw*, in de zin dat de afbeeldingen $a \to \varphi_A(a\,\cdot)$ en $a \to \varphi_A(\cdot\, a)$ beiden A injectief inbedden in de duale vectorruimte voor A, en φ_A is *modulair*, in de zin dat er een automorfisme σ_A van A bestaat zodat

$$\varphi_A(a\sigma_A(a')) = \varphi_A(a'a)$$

voor alle $a, a' \in A$. *Ten derde* is er ook een (niet-triviale) *rechts* invariante functionaal ψ_A aanwezig, i.e. een functionaal zodat

$$(\psi_A \otimes \iota_A)\Delta_A(a) = \psi_A(a)1_A$$

[4]waarbij men zin geeft aan deze identiteit door de u.u.e. eigenschap te gebruiken

[5]waarbij we echter opmerken dat de bijectiviteit van deze afbeeldingen niet allen onafhankelijk van elkaar zijn

voor alle $a \in A$. Deze functionaal is gemakkelijk te construeren: men stelt gewoon $\psi_A = \varphi_A \circ S_A$. Dan toont men aan dat ψ_A ook getrouw en modulair is. Er geldt echter meer: φ_A en ψ_A zijn nauw met elkaar verbonden door middel van een *modulair element* $\delta_A \in M(A)$: dit is een inverteerbare vermenigvuldiger van A zodat $\varphi_A(a\delta_A) = \psi_A(a)$ voor elke $a \in A$. *Ten slotte* kan er ook een scalaire invariant aan (A, Δ_A) verbonden worden: dit betreft het getal $\nu_A \in k$ zodat $\varphi_A \circ S_A^2 = \nu_A \cdot \varphi_A$. Natuurlijk bestaan er ook veel commutatie-relaties tussen al deze structuren.

Een andere mooie eigenschap van algebraïsche kwantumgroepen is dat ze een dualiteitstheorie toelaten, die nauw verwant is aan de Pontryagin dualiteit, gekend voor (lokaal compacte) abelse groepen. Inderdaad, gegeven een algebraïsche kwantumgroep A, dan kan men van de deelverzameling $\widehat{A} = \{\varphi_A(\,\cdot\, a) \mid a \in A\}$ van functionalen op A een algebraïsche kwantumgroep maken, door de structuur van A te transponeren (waarbij men de links invariant functionaal moet construeren met behulp van de co-eenheid van A). Er geldt dan Pontryagin dualiteit, in de zin dat de duale van de duale canoniek isomorf is met de oorspronkelijke algebraïsche kwantumgroep.

We weiden op het einde van het tweede hoofdstuk ook wat uit over een resultaat dat in [21] behaald werd. We moeten opnieuw eerst een definitie invoeren.

Definitie N.2.5. *Een* *-algebraïsche kwantumgroep *is een algebraïsche kwantumgroep over het veld* $\mathbb{C}$*, voorzien van een* **-structuur (i.e. een anti-multiplicatieve anti-lineaire involutie* **), zodat* $\Delta_A(a^*) = \Delta_A(a)^*$*, en zodat* $\varphi_A(a^*a) \geqslant 0$ *voor elke* $a \in A$.

In [21] tonen we dan aan hoe de verdere structuur van *-algebraïsche kwantumgroepen essentieel discreet van aard is: de algebra automorfismes σ_A en S_A^2 hebben positief puur puntspectrum (i.e., A heeft een basis van eigenvectoren voor deze automorfismes, en bovendien zijn alle eigenwaarden positief). Hetzelfde geldt voor het modulair element δ_A. Deze resultaten, die op betrekkelijk eenvoudige wijze bekomen kunnen worden, laten ons dan toe om significant eenvoudiger bewijzen te leveren voor enkele hoofdresultaten uit [53] en [55]. Het laat ons ook toe om te concluderen dat de scalaire invariant ν_A voor *-algebraïsche kwantumgroepen altijd triviaal 1 is (wat tot dan toe een open probleem was).

In het derde hoofdstuk behandelen we in detail de structuur van *Galois objecten* voor algebraïsche kwantumgroepen.

Definitie N.2.6. *Zij A een algebraïsche kwantumgroep. Een niet-ontaarde idempotente algebra B, samen met een u.u.e. homomorfisme $\alpha_B : B \rightarrow M(B \odot A)$, wordt een* (rechts) Galois object *voor A genoemd, als α_B een coactie is, i.e.*

$$(\alpha_B \otimes \iota_A)\alpha_B = (\iota_B \otimes \Delta_A)\alpha_B$$

en

$$\alpha_B(B)(1 \otimes A) = B \odot A = (1 \otimes A)\alpha_B(B),$$

en de afbeelding

$$G : B \odot B \rightarrow M(B \odot A) : b \otimes b' \rightarrow (b \otimes 1)\alpha_B(b'),$$

die we de Galois afbeelding *noemen, injectief is, met $B \odot A$ als beeld.*

Vooreerst construeren we dan twee speciale getrouwe functionalen op een Galois object B. De eerste functionaal, die we met φ_B noteren, is een δ_A-invariante functionaal, in de zin dat

$$(\varphi_B \otimes \iota_A)\alpha_B(b) = \varphi_B(b)\delta_A$$

voor alle $b \in B$. Deze functionaal is tamelijk direct te construeren: ze wordt volledig bepaald door de identiteit

$$(\iota_B \otimes \varphi_A)\alpha_B(b) = \varphi_B(b)1_B$$

voor alle $b \in B$. De tweede functionaal, die we met ψ_B zullen noteren, is niet zo canoniek te construeren, en zal dan ook slechts op een scalaire in k na bepaald zijn. Deze functionaal zal echter invariant zijn:

$$(\psi_B \otimes \iota_A)\alpha_B(b) = \psi_B(b)1_A$$

voor alle $b \in B$. We tonen dan aan dat, net als voor algebraïsche kwantumgroepen, deze twee functionalen met elkaar verbonden zijn door middel van een modulair element: een inverteerbare vermenigvuldiger δ_B van B die voldoet aan

$$\varphi_B(b\delta_B) = \psi_B(b)$$

voor alle $b \in B$.

Vervolgens tonen we aan dat zowel φ_B als ψ_B modulair zijn. Dit laat ons toe om een u.u.e. homomorfisme $\beta_A : A \to M(B^{\text{op}} \odot B)$ te definiëren, zodat

$$B^{\text{op}} \odot A \to B^{\text{op}} \odot B : b^{\text{op}} \otimes a \to \beta_A(a)(b^{\text{op}} \otimes 1)$$

een inverse voor de Galois afbeelding G bepaalt (na de canonieke identificatie van B^{op} en B als vectorruimtes). We merken op dat dit resultaat heel wat meer technische voorbereiding vraagt dan in het geval van Hopf algebra's!

Tenslotte komen we tot de merkwaardigste constructie aangaande Galois objecten, met name deze van een *antipode*. In feite construeren we eerst een *antipode kwadraat*, welke een automorfisme $S_B^2 : B \to B$ is. Zo'n antipode kwadraat werd ook geconstrueerd voor Hopf algebraïsche Galois objecten, al was het bestaan ervan niet meteen van in het begin duidelijk (zie bijvoorbeeld [75]). Onze constructiemethode is echter essentieel verschillend, en maakt gebruik van de aanwezige modulaire structuur. Eens deze antipode kwadraat er is, kunnen we *twee* antipodes construeren, die echter niet *intern* zijn: noteren we $C = B^{\text{op}}$, dan definiëren we de ene antipode S_C als de canonieke afbeelding

$$C \to B : b^{\text{op}} \to b,$$

terwijl we de tweede antipode S_B definiëren als

$$B \to C : b \to S_B^2(b)^{\text{op}}.$$

Deze twee afbeeldingen voldoen dan inderdaad aan de definiërende eigenschap van een antipode, maar tegenover de (externe) covermenigvuldiging β_A: noteren we formeel $\beta_A(a) = a_{[1]} \otimes a_{[2]} \in C \odot B$, dan geldt (opnieuw formeel)

$$S_C(a_{[1]})a_{[2]} = \varepsilon_A(a)1_B$$

en

$$a_{[1]}S_B(a_{[2]}) = \varepsilon_A(a)1_C,$$

voor elke $a \in A$.

We eindigen dit hoofdstuk met het bestuderen van twee speciale situaties.

Ten eerste gaan we na wat er gebeurt als de algebraïsche kwantumgroep van een speciaal type is, hetzij compact (wat betekent dat de onderliggende algebra een eenheid heeft), hetzij discreet (wat essentieel betekent dat de duale compact is, al zijn er ook intrinsiekere karakterisaties voorhanden). De algebra van een bijhorend Galois object blijkt dan precies van dezelfde vorm te zijn als de algebra van de kwantumgroep: voorzien van een eenheid als de bijhorende kwantumgroep compact is, en discreet (i.e. met elk principaal links of rechts ideaal eindig-dimensionaal) als de bijhorende kwantumgroep discreet is.

Ten tweede definiëren we een notie van *-*Galois object* voor *-algebraïsche kwantumgroepen. Een *-Galois object (B, α_B) voor een *-algebraïsche kwantumgroep A is een Galois object, zodat de algebra B verder voorzien is van een *goede* *-structuur (in de zin dat $\sum_i b_i^* b_i = 0$ impliceert dat elke $b_i = 0$), en zodat α_B deze *-structuur bewaart. We tonen dan aan dat φ_B en ψ_B positief zijn (mogelijk na vermenigvuldigen met een scalair getal), i.e., dat $\varphi_B(b^*b) \geqslant 0$ en $\psi_B(b^*b) \geqslant 0$ voor elke $b \in B$. We doen dit opnieuw door te tonen dat 'links (en rechts) vermenigvuldigen met het modulair element δ_B' een diagonaliseerbare lineaire afbeelding is, met enkel strikt positieve eigenwaardes.

In het vierde hoofdstuk introduceren we het begrip 'algebraïsche link kwantumgroepoïde'.

Definitie N.2.7. *Een* algebraïsche link kwantumgroepoïde *bestaat uit een drietal* (E, e, Δ_E)*, met* E *een niet-ontaarde algebra,* e *een idempotent in* $M(E)$ *die voldoet aan* $EeE = E$ *en* $E(1_E - e)E = E$*, en* Δ_E *een coassociatief u.e. homomorfisme* $E \to M(E \odot E)$ *dat voldoet aan*

$$\Delta_E(e) = e \otimes e$$

en

$$\Delta_E(1_E - e) = (1_E - e) \otimes (1_E - e),$$

zó *dat* $A := eEe$ *en* $D := (1_E - e)E(1_E - e)$*, samen met de beperking van* Δ_E*, algebraïsche kwantumgroepen worden.*

We tonen aan dat ook deze objecten voorzien zijn van co-eenheid en antipode, zodat ze zich tot zwakke link Hopf algebra's verhouden als algebraïsche kwantumgroepen tot Hopf algebra's. Vervolgens gaan we, via dualisatie, vanuit een rechts Galois object B voor een algebraïsche kwantumgroep A, een algebraïsche link kwantumgroepoïde bouwen. Dit is opnieuw een tamelijk technisch proces. Vooreerst gaan we de functionaal φ_B naar een functionaal ψ_C op $C = B^{\mathrm{op}}$ overzetten, door

$$\psi_C(b^{\mathrm{op}}) = \varphi_B(b)$$

te definiëren, en noteren dan $\widehat{C} = \{\psi_C(\,\cdot\, c) \mid c \in C\}$. Nu kunnen we de ruimte van functionalen $\widehat{B} = \{\varphi_B(\,\cdot\, b) \mid b \in B\}$ op B tot een rechtse $\widehat{A}$-module maken, door de coactie α_B te transponeren. We kunnen $\widehat{B}$ dan als lineaire afbeeldingen van $\widehat{A}$ naar $\widehat{B}$ beschouwen, via 'links vermenigvuldigen'. Anderzijds kan de ruimte $\widehat{C}$ geïdentificeerd worden met lineaire afbeeldingen van $\widehat{B}$ naar $\widehat{A}$, door de formule

$$(\omega_{21} \cdot \omega_{12})(a) = (\omega_{21} \otimes \omega_{12})\beta_A(a)$$

voor $\omega_{21} \in \widehat{C}, \omega_{12} \in \widehat{B}$ en $a \in A$. Als we dan $\widehat{D}$ definiëren als de lineaire span van de afbeeldingen van $\widehat{B}$ naar zichzelf, bekomen door eerst een element van $\widehat{C}$ toe te passen en dan een element van $\widehat{B}$, dan kunnen we al deze vectorruimtes samen groeperen in een directe som

$$\widehat{E} = \begin{pmatrix} \widehat{D} & \widehat{B} \\ \widehat{C} & \widehat{A} \end{pmatrix},$$

welke op natuurlijke wijze een algebra vormt (bijvoorbeeld als algebra van lineaire operatoren op de directe vectorruimte som $\begin{pmatrix} \widehat{B} \\ \widehat{A} \end{pmatrix}$). Dit levert ons de onderliggende algebra van de te construeren algebraïsche link kwantumgroepoïde. De covermenigvuldiging wordt dan bekomen door de vermenigvuldiging op B te transponeren tot een 'covermenigvuldiging[6]' op $\widehat{B}$, en deze op natuurlijke wijze uit te breiden tot $\widehat{E}$.

We zijn nu echter nog niet klaar: we willen immers dat $\widehat{D}$, die ondertussen een 'algebra met covermenigvuldiging' is, ook een links invariante functionaal bezit. We passen hiertoe de methode toe uit [23] (onze methode uit [19] was iets omslachtiger): door een gepaste lineaire bijectie op B te transponeren bekomen we een afbeelding $\sigma_{\widehat{B}} : \widehat{B} \to \widehat{B}$, die ons toelaat om op $\widehat{D}$ een functionaal $\varphi_{\widehat{D}}$ te definiëren via de formule

$$\varphi_{\widehat{D}}(\omega_{12} \cdot \omega_{21}) := \varphi_{\widehat{A}}(\omega_{21}\sigma_{\widehat{B}}(\omega_{12}))$$

[6]Men moet enige voorzichtigheid aan de dag leggen hieromtrent, daar $\widehat{B}$ geen algebra is, en er dus in het algemeen geen 'ruimte van vermenigvuldigers' is. Bijgevolg is het niet duidelijk waar de covermenigvuldiging terecht moet komen. Echter, omdat $\widehat{B}$ een rechtse A-module is, is er *wel* een natuurlijke beeldruimte van '(linkse) vermenigvuldigers' voorhanden.

voor $\omega_{12} \in \widehat{B}$ en $\omega_{21} \in \widehat{C}$. Een laatste computationeel technisch bewijs toont dan aan dat deze functionaal inderdaad links invariant is ten opzichte van de geconstrueerde covermenigvuldiging op $\widehat{D}$, zodat deze laatste een algebraïsche kwantumgroep is.

In een volgende sectie tonen we aan hoe we terug moeten, i.e. hoe we vanuit een algebraïsche link kwantumgroepoïde een Galois object kunnen construeren. Dit gebeurt in essentie opnieuw door alle structuur te dualiseren, en deze stap is niet zo moeilijk meer. Omdat een algebraïsche link kwantumgroepoïde echter *twee* algebraïsche kwantumgroepen met zich meedraagt, wier rollen volledig symmetrisch zijn, kunnen we niet één, maar *twee* Galois objecten maken. Deze kunnen dan gecombineerd worden in een bi-Galois object.[7] Dit betekent dus dat we Schauenburgs reconstructieproces, met een omweg via dualiteit, bewerkstelligd hebben voor algebraïsche kwantumgroepen:

Stelling N.2.8. *Zij A een algebraïsche kwantumgroep, en (B, α_B) een rechts A-Galois object. Dan bestaat er een algebraïsche kwantumgroep D en een linkse coactie γ_B van D op B, zodat (B, γ_B, α_B) een D-A-bi-Galois object is.*

We schetsen in de thesis ook kort hoe men aan kan tonen dat D en γ_B uniek bepaald zijn, al geven we hier geen volledig bewijs voor. We noemen D dan de gereflecteerde algebraïsche kwantumgroep (van A langsheen B).

We komen ook nog even terug op het geval van *-Galois objecten voor *-algebraïsche kwantumgroepen. In dit geval kunnen we namelijk aantonen dat de gereflecteerde algebraïsche kwantumgroep *ook* een *-algebraïsche kwantumgroep is, i.e. dat er een natuurlijke *-structuur bestaat die de links invariante functionaal positief maakt. We kunnen dit dan gebruiken om te tonen dat, naast het modulair element, ook de antipode kwadraat en het modulair automorfisme van een *-Galois object diagonaliseerbaar zijn, met positieve eigenwaardes.

In een laatste sectie behandelen we een concreet voorbeeld van een Galois object. We geven toe dat dit voorbeeld niet zo geschikt is om de algemene theorie te presenteren: het betreft immers een Galois object voor een Hopf algebra (met invariante functionaal), en past dus volledig binnen Schauen-

[7]De definitie van een bi-Galois object voor algebraïsche kwantumgroepen is volledig dezelfde als voor Hopf algebra's.

burgs theorie van (bi-Galois) objecten. Niettemin kunnen we in dit voorbeeld concreet nagaan hoe de dualiteitstheorie werkt. Het blijkt ook dat de gereflecteerde algebraïsche kwantumgroep in dit geval een *nieuwe familie van Hopf algebra's met invariante functionalen* oplevert. We zijn ons niet bewust van het voorkomen van deze voorbeelden in de literatuur, al is het best mogelijk dat ze een onderdeel vormen van een grotere familie die reeds bekend was.

N.3 von Neumann algebraïsche Galois objecten

We gaan nu over tot het tweede deel van de thesis, dat in het kader van de operatoralgebra's, en specifieker, von Neumann algebra's plaatsvindt.

In de eerste drie hoofdstukken van dit deel (hoofdstukken 5 tot en met 7) wordt een theorie van Galois objecten voor von Neumann algebraïsche kwantumgroepen ontwikkeld.

We beginnen dit deel met een hoofdstuk (hoofdstuk 5) dat een overzicht geeft aangaande von Neumann algebra's en hun gewichtentheorie.

Definitie N.3.1. *Een* von Neumann algebra *(of* W*-algebra*) is een unitale* **-algebra die isomorf is met een* σ*-zwak gesloten deel-***-algebra van de ruimte* $B(\mathscr{H})$ *van begrensde operatoren op* een *Hilbertruimte* $\mathscr{H}$.

De theorie van von Neumann algebra's, wier grondslagen reeds in de jaren '30 door Murray en von Neumann gelegd werden, is nog steeds een actief onderzoeksdomein, met verschillende subdisciplines (studie van II_1-factoren, studie van vrije probabiliteit, studie van deelfactoren, ...). Wij zullen vooral de structuurstellingen nodig hebben die eind jaren '60 door Tomita en Takesaki behaald werden, en die bekend staan onder de naam 'Tomita-Takesaki theorie'. Een overzicht van deze theorie is te vinden in de eerste hoofdstukken van het referentiewerk [84].

We presenteren eerst de definitie van een 'gewicht op een von Neumann algebra', wat een niet-commutatieve versie is van 'maat op een meetbare ruimte'.

Definitie N.3.2. *Zij* N *een von Neumann algebra. Een* gewicht *op* N *is een semi-lineaire afbeelding* φ *van de kegel van positieve elementen* N^+ *naar* *het onbegrensde interval* $[0, +\infty]$.

Een gewicht wordt getrouw *genoemd, als* $\varphi(x) = 0$ *voor een* $x \in N^+$ *impliceert dat* $x = 0$.

Een gewicht wordt semi-eindig *genoemd, als het linkse ideaal* $\mathscr{N}_\varphi$ *van elementen* $x \in N$ *waarvoor* $\varphi(x^*x) < \infty$ *een* σ*-dicht deel van* N *vormt.*

Een gewicht wordt normaal *genoemd, als voor elk stijgend naar boven begrensd net* $x_i \in N^+$ *geldt dat* $\varphi(x) = \lim \varphi(x_i)$*, waarbij* $x = \sup x_i$.

We zullen in het vervolg uitsluitend met normale, semi-eindige, getrouwe gewichten werken, en noemen deze dan *nsf gewichten* (waarbij we de afkorting van de *Engelse* termen blijven behouden). We merken op dat een nsf gewicht (beperkt en) uitgebreid kan worden tot een lineaire functionaal op elementen van de vorm $\sum_{i=1}^n x_i^* y_i$, met $x_i, y_i \in \mathscr{N}_\varphi$. Elementen van deze laatste vorm noemen we de *integreerbare elementen voor* φ, en we noteren de verzameling van al deze elementen met $\mathscr{M}_\varphi$.

De volgende stelling is een deel van het kernresultaat van Tomita-Takesaki-theorie, dat toont dat er op niet-commutatieve von Neumann algebra's een natuurlijke *'tijdsevolutie'* is.

Stelling N.3.3. *Zij* N *een von Neumann algebra, en* φ *een nsf gewicht op* N*. Dan bestaat er een* $\mathbb{R}$*-geparametrizeerde groep* σ_t^φ *van* **-automorfismes van* N*, de* modulaire één-parametergroep voor φ *genaamd, die op de volgende manier met* φ *verbonden is:* φ *is* σ_t^φ*-invariant, in de zin dat* $\varphi \circ \sigma_t^\varphi = \varphi$ *voor elke* $t \in \mathbb{R}$*, en voor elke* $x, y \in \mathscr{N}_\varphi \cap \mathscr{N}_\varphi^*$ *zal de volgende conditie gelden, de* KMS-conditie *genaamd*[8]*: er bestaat een begrensde analytische functie* $F_{x,y}$ *op het domein* $\{z \in \mathbb{C} \mid 0 < Im(z) < 1\}$*, uitbreidbaar tot een continue afbeelding op de sluiting van dit domein, zó dat* $F_{x,y}(t) = \varphi(\sigma_t^\varphi(y)x)$ *en* $F_{x,y}(t+i) = \varphi(x\sigma_t^\varphi(y))$ *voor alle* $t \in \mathbb{R}$.

We gebruiken in onze thesis vooral een variant van de KMS-conditie. Voor voldoende veel elementen $y \in N$ (die we in het vervolg 'elementen van de Tomita algebra van φ' zullen noemen) zal namelijk $t \to \sigma_t^\varphi(y)$ uit te breiden

[8]naar Kubo, Martin en Schwinger, die deze relatie ontdekten in verband met hun onderzoek omtrent statistische kwantummechanica

zijn tot een analytische (N-waardige) functie $z \in \mathbb{C} \to \mathscr{N}_\varphi \cap \mathscr{N}_\varphi^* : z \to \sigma_z^\varphi(y)$, die dan voldoet aan de volgende eigenschap: voor alle $x \in \mathscr{N}_\varphi \cap \mathscr{N}_\varphi^*$ zal

$$\varphi(yx) = \varphi(x\sigma_{-i}^\varphi(y)).$$

We zien dus dat het bestaan van de modulaire één-parametergroep ons toelaat om het 'spoorloze karakter' van φ op te vangen (waarbij we de lezer eraan herinneren dat een *spoor* op een algebra een functionaal τ is die voldoet aan $\tau(xy) = \tau(yx)$ voor alle x, y in de algebra).

Ook het volgende deel-resultaat van Tomita-Takesaki theorie wordt vaak in onze thesis gebruikt. We moeten echter eerst wat extra terminologie invoeren. Aan elk nsf gewicht kan een representatie van de von Neumann algebra verbonden worden. Dit heet *de GNS-constructie voor het nsf gewicht*, naar Gelfand, Naimark en Segal. Vooreerst creëert men met behulp van het gewicht de Hilbertruimte $\mathscr{L}^2(N,\varphi)$, bekomen door $\mathscr{N}_\varphi$ te completeren naar de norm

$$\|x\|_{\varphi,2} = \varphi(x^*x).$$

Deze voorziet men dan van de linkse representatie van N via links vermenigvuldigen. De canonieke afbeelding $\mathscr{N}_\varphi \to \mathscr{L}^2(N,\varphi)$ wordt verder genoteerd als Λ_φ. In het vervolg zullen we $\mathscr{L}^2(N,\varphi)$ met $\mathscr{L}^2(N)$ noteren, omdat men aan kan tonen dat er tussen alle linkse N-modules $\mathscr{L}^2(N,\varphi)$, met φ lopend over alle nsf gewichten, *natuurlijke* unitaire isomorfismes bestaan.

Stelling N.3.4. *Zij N een von Neumann algebra, en φ een nsf gewicht op N. De modulaire automorfismegroep voor φ wordt op $\mathscr{L}^2(N)$ geïmplementeerd door een canonieke één-parametergroep van unitairen ∇_φ^{it}, in de zin dat $\nabla_\varphi^{it} x \nabla_\varphi^{-it} = \sigma_t^\varphi(x)$ voor alle $x \in N$. We noemen de voortbrenger ∇_φ van deze één-parametergroep de* modulaire operator *voor φ.*

Er bestaat verder een anti-unitaire involutieve operator J_N op $\mathscr{L}^2(N)$, de modulaire conjugatie genaamd*, zodat J_N commuteert met ∇_φ^{it}, en zodat $J_N N J_N = N'$. Deze modulaire conjugatie is onafhankelijk van het gewicht φ.*

We stippen verder nog één resultaat aan uit dit hoofdstuk van onze thesis, dat in het verdere verloop van deze samenvatting nog even ter sprake zal komen. Dit heeft te maken met Morita theorie voor von Neumann algebra's.

Definitie N.3.5. *Zij M en P von Neumann algebra's. We noemen M en P* W*-Morita equivalent *als er een Hilbertruimte $\mathscr{H}$ bestaat, voorzien van een* *trouwe normale*[9] *unitale linkse* **-representatie van* P *en een trouwe normale rechtse* **-representatie van* M*,* zó *dat* P'*, de commutant van (het beeld van)* P *op* $\mathscr{H}$*, precies (het beeld van) de von Neumann algebra* M *is.*

De resulterende W*-Morita theorie is dan betrekkelijk eenvoudig, en kan op verschillende manieren gekarakteriseerd worden. Voor ons zal het echter van belang zijn om te weten of, en hoe, men een (nsf) gewicht op één van de von Neumann algebra's op *canonieke* wijze kan overdragen tot de andere von Neumann algebra. Een antwoord hierop wordt gegeven door het volgende resultaat van A. Connes.

Stelling N.3.6. *Zij* $\mathscr{H}$ *een Hilbertruimte, voorzien van een getrouwe rechtse normale unitale* **-representatie* θ *van een von Neumann algebra* M*. Zij* φ_M *een nsf gewicht op* M*, en veronderstel dat er op* $\mathscr{H}$ *een (*$\mathbb{R}$*-geparametriseerde) één-parametergroep van unitairen* ∇^{it} *bestaat, die* $\sigma_t^{\varphi_M}$ *op* M *implementeert:*

$$\nabla^{it}\theta(m)\nabla^{-it} = \theta(\sigma_t^{\varphi_M}(m)) \qquad \textit{voor alle } m \in M.$$

Dan kan men op canonieke wijze *een nsf gewicht* φ_P *op* $P = \theta(M)'$ *construeren, zodat* ∇^{it} *ook* $\sigma_t^{\varphi_P}$ *implementeert:*

$$\nabla^{it}x\nabla^{-it} = \sigma_t^{\varphi_P}(x) \qquad \textit{voor alle } x \in P.$$

Bovendien bestaat er voor elk nsf gewicht φ_P *op* P *een éénparametergroep van unitairen* ∇^{it} *op* $\mathscr{H}$ *die aan de bovenstaande conditie voldoet, zó dat de bovenstaande constructie precies* φ_P *oplevert. We noemen* ∇ *dan de* spatiale afgeleide van φ_P t.o.v. φ_M*, en noteren*

$$\nabla = \frac{d\varphi_P}{d\varphi_M'}.$$

We vermelden nog dat W*-Morita equivalentie ook geformuleerd kan met behulp van link structuren.

Definitie N.3.7. *Een* von Neumann link algebra *is een koppel* (Q, e) *bestaande uit een von Neumann algebra* Q *en een (zelftoegevoegde) projectie* $e \in Q$*, zodat de 2-zijdige idealen voortgebracht door* e *en* $1 - e$ *beiden* σ*-zwak dicht zijn in* Q*.*

Lemma N.3.1. *Zij* M *en* P *twee von Neumann algebra's. Dan zijn* M *en* P *W*-Morita equivalent als en slechts als er een von Neumann link algebra* (Q, e) *bestaat zodat* $M \cong eQe$ *en* $P \cong (1-e)Q(1-e)$*.*

[9] i.e. continu t.o.v. de σ-zwakke topologie

In het *zesde hoofdstuk* van onze thesis brengen we de belangrijkste resultaten uit de artikels [56], [57] en [85] samen. In de eerste twee van deze artikels wordt een elegante definitie van *lokaal compacte kwantumgroepen* voorgesteld. We geven enkel de von Neumann algebraïsche versie van deze definitie, die in [57] besproken wordt. Deze definitie is verbazend compact.

Definitie N.3.8. *Een* von Neumann algebraïsche kwantumgroep *bestaat uit een koppel* (M, Δ_M)*, waarbij* M *een von Neumann algebra is en* Δ_M *een unitaal normaal* **-homomorfisme* $M \to M \otimes M$ *is dat aan de coassociativiteitsvoorwaarde*

$$(\Delta_M \otimes \iota_M)\Delta_M = (\iota_M \otimes \Delta_M)\Delta_M$$

voldoet, en zó dat er nsf gewichten φ_M *en* ψ_M *op* M *bestaan, resp. het* links *en* rechts *invariante gewicht van de kwantumgroep genaamd, die voldoen aan de volgende eigenschap: voor elke* $\omega \in M_*^+$ *geldt dat*

$$\varphi_M((\omega \otimes \iota)\Delta_M(x)) = \varphi_M(x)\omega(1) \qquad \text{voor alle } x \in \mathscr{M}_{\varphi_M}^+,$$

en

$$\psi_M((\iota \otimes \omega)\Delta_M(x)) = \psi_M(x)\omega(1) \qquad \text{voor alle } x \in \mathscr{M}_{\psi_M}^+.$$

Hieruit wordt dan een rijke theorie ontwikkeld. In het bijzonder bestaat er bijvoorbeeld een geassocieerd C*-algebraïsch object (i.e. een 'niet-commutatieve topologische ruimte'), dat past binnen het kader van de C*-algebraïsche kwantumgroepen[10] die in [56] worden ingevoerd.

Puur formeel zijn we alle verdere structuren die voorkomen bij von Neumann algebraïsche kwantumgroepen reeds tegengekomen toen we de algebraïsche kwantumgroepentheorie uit de doeken deden. Alleen zullen de automorfismes die daar voorkomen nu veranderd worden in één-parametergroepen van automorfismes. Zo komt het modulair automorfisme σ_A voor de links invariante functionaal φ_A op een algebraïsche kwantumgroep A nu overeen met de modulaire één-parametergroep van automorfismes $\sigma_t^{\varphi_M}$ voor het links invariante gewicht φ_M op een von Neumann algebraïsche kwantumgroep M (en σ_A kan geïnterpreteerd worden als $\sigma_{-i}^{\varphi_M}$). Verder zal er een één-parametergroep τ_t^M van automorfismes op M zijn, de *schaalgroep* genaamd, zodat τ_{-i}^M correspondeert met de antipode kwadraat op een algebraïsche

[10] We willen hierbij opmerken dat het begrip *lokaal compacte kwantumgroep*, waar we op bepaalde plaatsen gebruik van hebben gemaakt, op zich niet echt bestaat: het is eerder een verzamelnaam voor alle C*-algebraïsche kwantumgroepen die eenzelfde geassocieerde von Neumann algebraïsche kwantumgroep hebben.

kwantumgroep. Samen met een zeker involutief anti-automorfisme R_M, de *unitaire antipode* genaamd, kunnen we dan op M een *antipode* $S_M = R_M \circ \tau^M_{-i/2}$ maken, welke nu echter geen katje is om zonder handschoenen aan te pakken: dit is immers een onbegrensde afbeelding van een (dicht deel van) M naar zichzelf. Hoewel hiermee dan inderdaad zin gegeven kan worden aan de antipode eigenschap, bekend van de Hopf algebra theorie, zullen we meestal teruggrijpen naar R_M en τ^M_t afzonderlijk. Verder is er voor een von Neumann algebraïsche kwantumgroep ook een *modulair element* δ_M voorhanden, dat nu een aan M geaffilieerde, onbegrensde, strikt positieve operator is, en een zekere scalaire invariant $\nu_M \in \mathbb{R}^+$, de *schaalconstante* genaamd. In [94] wordt een voorbeeld geconstrueerd waar deze schaalconstante niet triviaal is (in tegenstelling dus met de situatie voor *-algebraïsche kwantumgroepen, welke in feite een speciale klasse van von Neumann algebraïsche kwantumgroepen uitmaken).

Er is nog één verdere structuur die vermeld moet worden. Deze trad ook al impliciet op bij de algebraïsche kwantumgroepen.

Definitie N.3.9. *Zij $\mathscr{H}$ een Hilbertruimte. Een* multiplicatieve unitaire *op $\mathscr{H}$ is een unitaire $W \in B(\mathscr{H} \otimes \mathscr{H})$, die voldoet aan de pentagon-gelijkheid:*

$$W_{12}W_{13}W_{23} = W_{23}W_{12}.$$

Hierbij hebben we gebruik gemaakt van de beentjesnummering: W_{ij} is de operator op een tensorproduct van een willekeurig aantal kopieën van $\mathscr{H}$, die op de i-de en j-de component als W werkt, en de overige componenten ongemoeid laat. Bijvoorbeeld: W_{12} op $\mathscr{H} \otimes \mathscr{H} \otimes \mathscr{H}$ is gewoon de operator $W \otimes 1$. De notie van een multiplicatieve unitaire (en van bijhorende regulariteitseisen) werd ontwikkeld in het artikel [4].

Voor elke von Neumann algebraïsche kwantumgroep is er canoniek zo een multiplicatieve unitaire W beschikbaar.

Definitie N.3.10. *Zij (M, Δ_M) een von Neumann algebraïsche kwantumgroep met links invariant nsf gewicht φ_M. Dan bestaat er een unieke unitaire $W_M \in M \otimes B(\mathscr{L}^2(M))$, de* links reguliere corepresentatie *genaamd, zodat voor elke $\omega \in B(\mathscr{L}^2(M))_*$ en $x \in \mathscr{N}_{\varphi_M}$ geldt, dat $(\omega \otimes \iota)\Delta_M(x) \in \mathscr{N}_{\varphi_M}$ en*

$$(\omega \otimes \iota)(W_M^*)\Lambda_{\varphi_M}(x) = \Lambda_{\varphi_M}((\omega \otimes \iota)\Delta(x)).$$

Deze unitaire W_M is dan een multiplicatieve unitaire.

We merken op dat het tamelijk gemakkelijk is om aan te tonen dat W_M^* een isometrie is die aan de pentagon-gelijkheid voldoet. Wat helemaal niet triviaal is, is het bewijs van de surjectiviteit van W_M^*. Dit vormt één van de mooie maar heel technische constructies uit [56].

Met behulp van de multiplicatieve unitaire kan men dan een dualiteitstheorie voor von Neumann algebraïsche kwantumgroepen ontwikkelen. Hierbij definieert men de onderliggende von Neumann algebra $\widehat{M}$ als de σ-zwakke sluiting van de verzameling $\{(\omega\otimes\iota)(W_M) \mid \omega \in M_*\}$ (waarvan men kan tonen dat het inderdaad een von Neumann algebra vormt). De covermenigvuldiging wordt gedefinieerd met behulp van W_M: noteren we $W_{\widehat{M}} = \Sigma W_M^* \Sigma$, dan stellen we

$$\Delta_{\widehat{M}}(x) = W_{\widehat{M}}^*(1 \otimes x)W_{\widehat{M}} \qquad \text{voor alle } x \in \widehat{M}.$$

We vermelden dat er ook een *rechts* reguliere corepresentatie V_M van (M, Δ_M) bestaat. Dit is dan een multiplicatieve unitaire die in $\widehat{M}' \otimes M$ gelegen is.

Spreekt men over groepen, dan moet men het ook over hun bijhorende acties en representaties hebben. In [85] wordt in detail besproken hoe men een theorie van coacties[11] van von Neumann algebraïsche kwantumgroepen op von Neumann algebra's kan ontwikkelen.

Definitie N.3.11. *Zij N een von Neumann algebra, en (M, Δ_M) een von Neumann algebraïsche kwantumgroep. Een* rechtse coactie van M op N *bestaat uit een trouw normaal unitaal *-homomorfisme $\alpha : N \to N \otimes M$, zodat*

$$(\alpha \otimes \iota)\alpha = (\iota \otimes \Delta_M)\alpha.$$

Er bestaan ook natuurlijke, niet-commutatieve veralgemeningen van 'acties met speciale eigenschappen'. De volgende definitie verschaft twee voorbeelden hiervan.

Definitie N.3.12. *Zij N een von Neumann algebra, M een von Neumann algebraïsche kwantumgroep, en α een rechtse coactie van M op N.*

Men noemt α ergodisch*, als enkel de scalaire veelvouden van de eenheid in N voldoen aan de vergelijking*

$$\alpha(x) = x \otimes 1.$$

[11] In [85] wordt over acties gesproken - wij zullen het hebben over coacties.

Men noemt α integreerbaar *als er een* σ*-zwak dicht deel* $\mathscr{M}_{T_\alpha}$ *van* N *bestaat, zodat* $(\omega \otimes \iota)\alpha(x) \in \mathscr{M}_{\varphi_M}$ *voor alle* $\omega \in N_*$ *en* $x \in \mathscr{M}_{T_\alpha}$.

Aan elke coactie kan verder ook een nieuwe von Neumann algebra geassocieerd worden, welke men het gekruist product noemt.

Definitie N.3.13. *Zij* N *een von Neumann algebra,* M *een von Neumann algebraïsche kwantumgroep, en* α *een coactie van* M *op* N*. Dan noemt men de bicommutant van de verzameling van operatoren* $\alpha(N) \cup (1 \otimes \widehat{M}')$ *op* $\mathscr{L}^2(N) \otimes \mathscr{L}^2(M)$ *het* gekruist product van N met M*. We noteren deze von Neumann algebra als* $N \underset{\alpha}{\rtimes} M$*, of gewoon* $N \rtimes M$ *als* α *duidelijk is uit de context.*

Nu voeren we het begrip *unitaire corepresentatie voor een von Neumann algebraïsche kwantumgroep* in.

Definitie N.3.14. *Zij* M *een von Neumann algebraïsche kwantumgroep, en* $\mathscr{H}$ *een Hilbertruimte. Een* (rechtse) unitaire corepresentatie van M op $\mathscr{H}$ *is een unitaire* $U \in B(\mathscr{H}) \otimes M$ *die voldoet aan de vergelijking*

$$(\iota \otimes \Delta)U = U_{12}U_{13}.$$

De volgende stellingen zijn twee van de mooie resultaten uit [85]. De eerste is een veralgemening van een stelling van Haagerup.

Stelling N.3.15. *Zij* N *een von Neumann algebra,* M *een von Neumann algebraïsche kwantumgroep, en* α *een coactie van* M *op* N*. Dan kan men* canoniek *een unitaire rechtse corepresentatie* U *op* $\mathscr{L}^2(N)$ *construeren die de coactie implementeert:*

$$U(x \otimes 1)U^* = \alpha(x)$$

voor alle $x \in N$.

Men noemt U dan de *unitaire implementatie* van α.

Stelling N.3.16. *Zij* N *een von Neumann algebra,* M *een von Neumann algebraïsche kwantumgroep, en* α *een coactie van* M *op* N*. Zij* U *de unitaire implementatie van* α*. Dan is* α *integreerbaar* als en slechts als *er een normaal* **-homomorfisme* $\rho_\alpha : N \rtimes M \to B(\mathscr{L}^2(N))$ *bestaat zodat*

$$\rho_\alpha(\alpha(x)) = x \qquad \text{voor alle } x \in N,$$

$$\rho_\alpha((1 \otimes (\iota \otimes \omega)(V_M))) = (\iota \otimes \omega)(U) \qquad \text{voor alle } \omega \in M_*.$$

In het geval α een integreerbare coactie is, noemen we het homomorfisme ρ_α uit deze stelling *het Galois homomorfisme voor* α. Beperken we ρ_α tot $\widehat{M}' \cong 1 \otimes \widehat{M}' \subseteq N \rtimes M$, dan bekomen we een normale linkse representatie $\widehat{\pi}'_\alpha$ van $\widehat{M}'$ op $\mathscr{L}^2(N)$, en bijgevolg ook een normale rechtse representatie $\widehat{\theta}_\alpha$ van $\widehat{M}$ op $\mathscr{L}^2(N)$ via de formule $\widehat{\theta}_\alpha(x) = \widehat{\pi}'_\alpha(J_{\widehat{M}} x^* J_{\widehat{M}})$.

We kunnen na deze voorbereidingen nu de definitie geven van een Galois object in de context van von Neumann algebraïsche kwantumgroepen.

Definitie N.3.17. *Zij N een von Neumann algebra, M een von Neumann algebraïsche kwantumgroep, en α een integreerbare coactie van M op N. We noemen de coactie* Galois *als het Galois homomorfisme trouw (i.e. injectief) is. We noemen (N, α) een* Galois object *indien α zowel Galois als ergodisch is.*

In het zevende hoofdstuk van onze thesis bestuderen we dan in detail de verdere structuur van Galois objecten. De bekomen resultaten zijn oppervlakkig gelijkend aan deze die voor de algebraïsche kwantumgroepen behaald werden, maar vergen wat meer technische finesse.

Eerst merken we op dat een Galois object (N, α) canoniek van een nsf gewicht voorzien kan worden: voor $x \in M^+_{T_\alpha}$ bestaat namelijk, wegens ergodiciteit, een positief getal $\varphi_N(x)$ zodat

$$\varphi_N(x) = \varphi_M((\omega \otimes \iota)(\alpha(x)))$$

voor elke normale toestand ω op N. Als we verder $\varphi_N(x) = +\infty$ definiëren voor $x \in N^+ \backslash M^+_{T_\alpha}$, dan wordt φ_N een nsf gewicht op N, waarbij het semifiniet zijn volgt uit het integreerbaar zijn van de coactie.

We kunnen voor een Galois object een analytische variant van de Galois afbeelding voor algebraïsche Galois objecten maken. Dit betreft nu een unitaire afbeelding $\tilde{G}$ van $\mathscr{L}^2(N) \otimes \mathscr{L}^2(N)$ naar $\mathscr{L}^2(M) \otimes \mathscr{L}^2(N)$, de Galois unitaire genaamd, uniek bepaald door de formule

$$(\iota \otimes \omega)(\tilde{G})\Lambda_{\varphi_N}(x) = \Lambda_{\varphi_M}((\omega \otimes \iota)\alpha_N(x))$$

voor alle $x \in \mathscr{N}_{\varphi_N}$ en $\omega \in B(\mathscr{L}^2(N))$.

Vervolgens maken we een één-parametergroep τ_t^N van *-automorfismes op het Galois object, die dan dezelfde rol speelt als de schaalgroep voor een von Neumann algebraïsche kwantumgroep, en die we dus dezelfde naam zullen toebedelen. Deze één-parametergroep wordt als volgt geconstrueerd. Zij $\delta_{\widehat{M}}$ het modulaire element van de duale kwantumgroep $\widehat{M}$. Dan wordt aangetoond dat $\widehat{\theta}_\alpha(\delta_{\widehat{M}}^{is})$ en $\nabla_{\varphi_N}^{it}$ commuteren. Bijgevolg geeft dit ons een één-parametergroep van unitairen $P_N^{it} = \widehat{\theta}_\alpha(\delta_{\widehat{M}}^{it})\nabla_{\varphi_N}^{it}$ op $\mathscr{L}^2(N)$. Deze implementeren dan de schaalgroep τ_t^N op N:

$$\tau_t^N(x) = P_N^{it} x P_N^{-it} \qquad \text{voor } x \in N.$$

(Deze constructie wordt in feite reeds ingevoerd in het zesde hoofdstuk van onze thesis, in de algemenere setting van integreerbare acties. De schaalgroep is daar echter niet canoniek, omdat er geen canoniek gewicht φ_N is.)

We kunnen ook een modulair element δ_N aan een Galois object associëren. We gaan de constructie ervan hier niet in detail verder bespreken, maar geven enkel de essentiële stappen aan. Eerst wordt de modulaire operator δ_M van M overgebracht op $\mathscr{L}^2(N) \otimes \mathscr{L}^2(N)$ via de Galois unitaire $\tilde{G}$. We tonen dan aan dat de geassocieerde één-parametergroep van automorfismes op $B(\mathscr{L}^2(N) \otimes \mathscr{L}^2(N))$ zich beperkt tot een één-parametergroep van automorfismes op $B(\mathscr{L}^2(N)) \cong 1 \otimes B(\mathscr{L}^2(N))$. Maar zo een één-parametergroep wordt noodzakelijk geïmplementeerd door een unitaire één-parametergroep op $\mathscr{L}^2(N)$. De voortbrenger hiervan levert dan het modulaire element δ_N, die op een scalaire na bepaald zal zijn.

We tonen tenslotte aan dat de operatoren P_N en $\theta_N(\delta_N)$ sterk commuteren, waarbij θ_N de canonieke rechtse representatie is van N op $\mathscr{L}^2(N)$, gegeven door $\theta_N(x) = J_N x^* J_N$. Bijgevolg kunnen we een één-parametergroep

$$\nabla_{\hat{N}}^{it} = P_N^{it} \theta_N(\delta_N^{-it})$$

definiëren, die in het vervolg de belangrijkste rol zal spelen: het blijkt namelijk dat

$$\nabla_{\hat{N}}^{it} \widehat{\theta}_\alpha(x) \nabla_{\hat{N}}^{-it} = \widehat{\theta}_\alpha(\sigma_t^{\varphi_M})$$

voor alle $x \in \widehat{M}$, zodat we Stelling N.3.6 kunnen toepassen en zo op canonieke wijze een nsf gewicht $\varphi_{\widehat{P}}$ op $\widehat{P} := \widehat{\theta}_\alpha(\widehat{M})'$ bekomen.

We gaan nu over tot de reflectietechniek in de context van Galois objecten voor von Neumann algebraïsche kwantumgroepen: we construeren op $\widehat{P}$ de structuur van een von Neumann algebraïsche kwantumgroep. De covermenigvuldiging wordt geïmplementeerd door $\tilde{G}$: de operatie

$$\Delta_{\widehat{P}} : \widehat{P} \to \widehat{P} \otimes \widehat{P} : x \to \tilde{G}^*(1 \otimes x)\tilde{G}$$

is goedgedefinieerd en coassociatief. Het blijkt verder ook voldoende te zijn om een links invariant nsf gewicht op $\widehat{P}$ te vinden, omdat we eenvoudig een 'unitaire antipode' op $\widehat{P}$ kunnen maken via de formule

$$R_{\widehat{P}}(x) = J_N x^* J_N \qquad \text{voor } x \in \widehat{P}.$$

Maar dit links invariante gewicht blijkt nu niets anders te zijn dan het gewicht $\varphi_{\widehat{P}}$ dat in de vorige paragraaf geconstrueerd werd.

In de verdere secties van het zevende hoofdstuk leggen we het verband tussen Galois objecten en de theorie van de von Neumann algebraïsche kwantumgroepoïdes (measured quantum groupoids), ontwikkeld in [59]. Zij namelijk N opnieuw een rechts Galois object voor een von Neumann algebraïsche kwantumgroep M. Zij $\widehat{N}$ de verzameling van begrensde operatoren $x : \mathscr{L}^2(M) \to \mathscr{L}^2(N)$ die voldoen aan

$$x\theta_{\widehat{M}}(y) = \widehat{\theta}_\alpha(y)x \qquad \text{voor alle } y \in \widehat{M},$$

waarbij $\theta_{\widehat{M}}$ de natuurlijke rechtse representatie van $\widehat{M}$ op $\mathscr{L}^2(M)$ is. Zij $\widehat{O} = \widehat{N}^*$, en verder $\widehat{P}$ als in de vorige paragraaf. Dan kunnen we de von Neumann algebra $\widehat{Q} = \begin{pmatrix} \widehat{P} & \widehat{N} \\ \widehat{O} & \widehat{M} \end{pmatrix}$ vormen, werkende op $\begin{pmatrix} \mathscr{L}^2(N) \\ \mathscr{L}^2(M) \end{pmatrix}$. Deze is op natuurlijke wijze een von Neumann link algebra. We weten verder dat op $\widehat{P}$ en $\widehat{M}$ een covermenigvuldiging aanwezig is. Maar we kunnen ook een covermenigvuldiging $\widehat{N} \to \widehat{N} \otimes \widehat{N}$ maken, gegeven door de formule

$$\Delta_{\widehat{N}}(x) = \tilde{G}^*(1 \otimes x)W_{\widehat{M}}, \qquad \text{voor alle } x \in \widehat{M},$$

waarbij $W_{\widehat{M}}$ de links reguliere corepresentatie voor $\widehat{M}$ is. Analoog kan een covermenigvuldiging $\widehat{O} \to \widehat{O} \otimes \widehat{O}$ gevormd worden, en deze kunnen dan allen gebundeld worden in een covermenigvuldiging $\Delta_{\widehat{Q}} : \widehat{Q} \to \widehat{Q} \otimes \widehat{Q}$, waarbij we echter opmerken dat deze laatste afbeelding niet eenheidsbewarend zal zijn. Het koppel $(\widehat{Q}, \Delta_{\widehat{Q}})$ blijkt dan, na een kleine aanpassing die in hoofdstuk 11 uitgewerkt wordt, binnen het formalisme van [58] te passen. We kunnen de koppels $(\hat{Q},\Delta_{\hat{Q}})$ die optreden ook abstract karakteriseren, en noemen deze *von Neumann algebraïsche link kwantum groepoïdes.*

Ook het duale concept, namelijk dit van een von Neumann algebraïsche *co*-link kwantumgroepoïde, kan abstract gekarakteriseerd worden. Ditmaal betreft het een directe som

$$Q = P\oplus O\oplus N\oplus M$$

van von Neumann algebra's, voorzien van een covermenigvuldiging Δ_Q : $Q\rightarrow Q\otimes Q$, zó dat, als we $P\oplus O\oplus N\oplus M$ als $Q_{11}\oplus Q_{21}\oplus Q_{12}\oplus Q_{22}$ schrijven, Δ_Q onder andere voldoet aan $\Delta_Q(Q_{ij})\subseteq\sum_{k=1}^2 Q_{ik}\otimes Q_{kj}$. Deze laatste conditie is duaal aan de matrix-vermenigvuldiging van 2-bij-2-matrices. Schrijven we Δ_{ij}^k voor de covermenigvuldiging Δ_Q met bron beperkt tot Q_{ij} en beeld tot $Q_{ik}\otimes Q_{kj}$, dan vindt men dat voor von Neumann algebraïsche co-link kwantumgroepoïdes het koppel (M,Δ_{22}^2) een von Neumann algebraïsche kwantumgroepoïde is, en (N,Δ_{12}^2) een rechts Galois object voor M. Omgekeerd tonen we aan dat elk rechts Galois object vervolledigd kan worden tot een von Neumann algebraïsche *co*-link kwantumgroepoïde, essentieel door de constructie uit de vorige paragraaf toe te passen, en dit dan te dualiseren, gebruik makende van de theorie uit [59].

Als nu twee von Neumann algebraïsche kwantumgroepen de hoeken uitmaken van een von Neumann algebraïsche link kwantum groepoïde, dan noemen we ze *comonoïdaal* W*-Morita equivalent, en hun duales *monoïdaal W*-co-Morita equivalent.* We tonen natuurlijk aan in de thesis dat dit werkelijk een equivalentie-relatie bepaald. Dit wordt bewerkstelligd door te tonen dat er een natuurlijke compositie van von Neumann algebraïsche (co-)link kwantum groepoïdes voorhanden is.

In een laatste sectie tonen we dan aan dat er ook een geassocieerde C*-algebraïsche theorie is: gebruik makend van de resultaten die in het elfde hoofdstuk behaald worden, tonen we dat comonoïdale W*-Morita equivalentie tussen von Neumann algebraïsche kwantumgroepen leidt tot een 'comonoïdale C*-Morita equivalentie' tussen de geassocieerde C*-algebraïsche kwantumgroepen, zowel op gereduceerd als op universeel vlak. Bovendien kan ook het Galois object N zelf voorzien worden van C*-algebraïsche structuren, namelijk een gereduceerde C*-algebra $B\subseteq N$, en een universele C*-algebra $B^u\twoheadrightarrow B$.

N.4 Constructiemethodes

Ons achtste hoofdstuk behandelt vier natuurlijke constructiemethodes. Om deze uit te kunnen leggen, moeten we eerst het concept 'gesloten kwantum deelgroep' introduceren, dat in onze thesis in het zesde hoofdstuk aan bod komt.

Definitie N.4.1. *Zij* (M, Δ) *een von Neumann algebraïsche kwantumgroep. We noemen een koppel* (M_1, F) *een* gesloten kwantum deelgroep *van* M *als* (M_1, Δ_1) *een von Neumann algebraïsche kwantumgroep is, en* $F : M_1 \to M$ *een unitaal getrouw normaal* **-homomorfisme zodat* $(F \otimes F) \circ \Delta_1 = \Delta \circ F$.

Vaak zullen we M_1 gewoon identificeren met zijn beeld onder F, en de notatie F weglaten.

Ons eerste resultaat zegt dan dat een Galois coactie van een von Neumann algebraïsche kwantumgroep M op een von Neumann algebra N *beperkt* kan worden tot een Galois coactie van een von Neumann algebraïsche kwantumgroep M_1, gegeven dat $\widehat{M_1} \subseteq \widehat{M}$ een gesloten kwantum deelgroep is. Een tweede resultaat zegt dat we Galois objecten kunnen *reduceren*: nu is eerder $M_1 \subseteq M$ een gesloten kwantumdeelgroep, en we maken vanuit een Galois object N voor M een Galois object N_1 voor M_1. We tonen bovendien aan dat onder de reflectieconstructie, toegepast op de Galois objecten N en N_1, de inclusie $M_1 \subseteq M$ overgestuurd wordt op een inclusie $P_1 \subseteq P$ van kwantumgroepen.

Het derde resultaat uit dit hoofdstuk toont aan dat er een één-één-verband is tussen coacties van monoïdaal W*-co-Morita equivalente von Neumann algebraïsche kwantumgroepen, en dat onder deze bijectie het ergodisch, integreerbaar en Galois zijn van een coactie bewaard blijft. Een vierde resultaat tenslotte toont dat als $\widehat{M_1} \subseteq \widehat{M}$ een gesloten kwantum deelgroep is, we een Galois object N_1 voor M_1 kunnen *induceren* tot een Galois object N voor M, en dat de reflecties langsheen N_1 en N leiden tot een inclusie $\hat{P}_1 \subseteq \hat{P}$ van von Neumann algebraïsche kwantumgroepen.

N.5 Toepassingen: 2-cocykels en projectieve representaties

In de volgende twee hoofdstukken van onze thesis beschouwen we enkele toepassingen. In het negende hoofdstuk bestuderen we het speciale geval van 'cleft' Galois objecten: dit betreft Galois objecten geconstrueerd met behulp van 2-cocykels.

Definitie N.5.1. *Zij* (M, Δ) *een von Neumann algebraïsche kwantumgroep. Een* unitaire 2-cocykel *voor* M *is een unitair element* $\Omega \in M \bar{\otimes} M$ *dat voldoet aan de vergelijking*

$$(\Omega \otimes 1)(\Delta \otimes \iota)(\Omega) = (1 \otimes \Omega)(\iota \otimes \Delta)(\Omega).$$

Als nu zo'n 2-cocykel gegeven is, kunnen we gemakkelijk een nieuwe covermenigvuldiging op M construeren, namelijk

$$\Delta_\Omega(x) = \Omega \Delta(x) \Omega^*.$$

De 2-cocykel identiteit laat meteen zien dat dit een coassociatieve covermenigvuldiging oplevert. Het is evenwel niet duidelijk of dit opnieuw een von Neumann algebraïsche kwantumgroep zal opleveren, i.e. of er invariante gewichten beschikbaar zijn. Men kan echter aan Ω een Galois object voor $\widehat{M}$ associëren, en (M, Δ_Ω) blijkt dan niets anders te zijn dan de von Neumann algebraïsche kwantumgroep die bekomen wordt door M te reflecteren langsheen dit Galois object. Als 'toemaatje' classificeren we ook de Galois objecten voor 'directe producten van von Neumann algebraïsche kwantumgroepen' aan de hand van de Galois objecten voor de afzonderlijke factoren en de bikarakters tussen de twee factoren. In het bijzonder kan dit toegepast worden op de theorie van Galois objecten voor (veralgemeende) 'Drinfel'd doubles' van kwantumgroepen.

In het tiende hoofdstuk voeren we het begrip 'projectieve (co-)representatie' voor von Neumann algebraïsche kwantumgroepen in. In de klassieke theorie van lokaal compacte groepen is er namelijk een nauw verband tussen 2-cocykels op de groep enerzijds, en acties van de groep op type I-factoren anderzijds. Dit gaat als volgt: zij $\mathfrak{G}$ een lokaal compacte groep met aftelbare basis, en $\mathscr{H}$ een separabele Hilbertruimte. Als $\alpha : \mathfrak{G} \rightarrow \mathrm{Aut}(B(\mathscr{H}))$ een continu homomorfisme is, met $\mathrm{Aut}(B(\mathscr{H})$ voorzien van de puntsgewijs σ-zwakke topologie, dan kan men voor elke $g \in \mathfrak{G}$ een unitaire u_g op $\mathscr{H}$ vinden zodat $\alpha_g(x) = u_g x u_g^*$ voor elke $x \in B(\mathscr{H})$. Bovendien kan men er voor zorgen dat $g \rightarrow u_g$ meetbaar is. Er bestaat dan een meetbare functie $\Omega : \mathfrak{G} \times \mathfrak{G} \rightarrow S^1 \subseteq \mathbb{C}$, met S^1 de cirkelgroep, zodat

$$\Omega(g, h) u_{gh} = u_g u_h$$

voor alle $g, h \in \mathfrak{G}$. We kunnen Ω interpreteren als een element van $\mathscr{L}^\infty(\mathfrak{G}) \bar{\otimes} \mathscr{L}^\infty(\mathfrak{G})$, en dit is dan precies een 2-cocykel voor de von Neumann algebraïsche 'kwantum'-groep $\mathscr{L}^\infty(\mathfrak{G})$. (Merk op dat Ω niet eenduidig bepaald wordt door α. Zijn cohomologieklasse is dit echter wel.) We noemen $g \to u_g$ dan een Ω-*representatie*, en, als Ω niet van tevoren gespecifieerd is, een *projectieve representatie* van $\mathfrak{G}$. Anderzijds levert elke projectieve representatie $g \to u_g$ gemakkelijk een actie op $B(\mathscr{H})$ op, door $\alpha_g(x) = u_g x u_g^*$ te stellen.

In de kwantumcontext gaat het verband tussen coacties op type I factoren en projectieve corepresentaties nog steeds op, mits men 2-cocykels vervangt door de meer algemene Galois objecten. Het begrip projectieve corepresentatie moet nu als volgt geïnterpreteerd worden.

Definitie N.5.2. *Zij (M, Δ) een von Neumann algebraïsche kwantumgroep, en (N, α) een Galois object voor M. Een projectieve linkse N-corepresentatie van M op een Hilbertruimte $\mathscr{H}$ is een unitair element $U \in \hat{N} \otimes B(\mathscr{H})$ dat voldoet aan*

$$(\Delta_{\hat{N}} \otimes \iota)U = U_{13}U_{23}.$$

Er blijkt dan inderdaad te gelden dat, als een coactie van een von Neumann algebraïsche kwantumgroep $\widehat{M}$ op een factor $B(\mathscr{H})$ gegeven is, we hier een Galois object N voor M aan kunnen associëren, samen met een projectieve linkse N-corepresentatie op $\mathscr{H}$. We tonen verder aan dat projectieve N-corepresentaties in één-één-verband gebracht kunnen worden met niet-ontaarde rechtse *-representaties van B^u, de universele C*-algebra geassocieerd aan N.

We buiten dit verband dan uit in het specifieke geval dat $\widehat{M}$ een *compacte* kwantumgroep is, i.e. voorzien is van *eindige* invariante gewichten (zodat $\varphi_{\widehat{M}}(1) = 1$).

In het klassieke geval van compacte groepen kan aangetoond worden dat irreducibele projectieve representaties noodzakelijk eindig dimensionaal zijn. In het kwantumgeval blijkt dit niet langer waar, en we geven een expliciet voorbeeld van dit fenomeen (me aangereikt door Stefaan Vaes). Dit zorgt voor het volgende merkwaardige fenomeen: als men het Galois object beschouwd, geassocieerd aan een oneindig dimensionale irreducibele projectieve corepresentatie van een compacte kwantumgroep, dan zal de comonoïdaal W*-Morita equivalente von Neumann algebraïsche kwantumgroep, geassocieerd aan dit Galois object, niet langer compact zijn. Omdat we er ook voor kunnen zorgen dat het bijhorende Galois object cleft is, bekomen we het volgende merkwaardige resultaat: er bestaat een compacte kwantumgroep, voorzien van een 2-cocykel Ω, zodat de kwantumgroep met het Ω-getwiste coproduct niet langer compact is.

N.6 von Neumann algebraïsche kwantumgroepoïdes met eindige basis

Het elfde hoofdstuk van onze thesis tenslotte staat inhoudelijk wat apart van de rest van de thesis. Het betreft hier een tamelijk summiere uiteenzetting van de theorie van von Neumann algebraïsche kwantum groepoïdes met een eindige basis. De belangrijkste resultaten betreffen hier het construeren van geassocieerde gereduceerde en universele C*-algebraïsche structuren. De methodes zijn sterk geïnspireerd (en gelijkend aan) deze uit de artikels [105] en [54]. De reden tot het behandelen van deze kwesties is het feit dat dit toelaat om de C*-algebraïsche resultaten omtrent von Neumann algebraïsche link en co-link algebra's samen te behandelen.

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List of symbols

$(\mathscr{D}\psi : \mathscr{D}\varphi)_t$ cocycle derivative

(E, e) (co-)linking (weak (multiplier) Hopf) algebra

(N, γ_N) (left) Galois object

$(N_\Omega, \gamma_\Omega, \alpha_\Omega)$ bi-Galois object associated to 2-cocycle, page 295

(Q, e) (co-)linking von Neumann algebra(ic quantum groupoid)

σ_A modular automorphism ψ_A

σ_t^M modular automorphism group ψ_M

α right coaction

α_V, α_B right comodule, right coaction

β_A external comultiplication

$\circ$ composition of maps, composition of morphisms

δ_N modular element of N

Δ_A comultiplication

δ_A modular element

$\Delta_A^{(2)}$ $(\Delta_A \otimes \iota)\Delta_A$

Δ_{ij} constituents of comultiplication on linking structure

ϵ non-normalized Markov trace

η_A unit map of an algebra, page 18

$\frac{d\psi}{d\varphi'}$	spatial derivative
Γ	comultiplication of a measured quantum groupoid
γ, Υ	left coaction
$\gamma_t^{\widehat{M}'}$	$\mathrm{Ad}(q_M^{it})$ restricted to $\widehat{M}'$, page 208
Γ_M	scaled GNS map for ψ_M, page 186
γ_V, γ_B	left comodule, left coaction
ι	identity map, identity morphism
$\iota \otimes \varphi$	slice operator valued weight
κ_t^M	$\delta_M^{-it} \tau_{-t}^M(\,\cdot\,) \delta_M^{it}$
κ_A	$\sigma_A^{-1} \circ S_A^2$
λ	reduced left representation of $\mathscr{L}_*^1(Q)$, page 332
λ^u	universal left representation for $\mathscr{L}_*^1(Q)$, page 332
Λ_M	GNS map φ_M, page 187
Λ_T	KSGNS map w.r.t. operator valued weight T_2, page 172
Λ_φ	GNS map
$\Lambda_{\widehat{M}}$, $\widehat{\Lambda}_M$	semi-cyclic representation of $\widehat{M}$ in $\mathscr{L}^2(M)$, page 189
$\langle a, b \rangle_A$	inner product on $\widehat{A}$, page 85
$[\,\cdot\,]$	normclosure of linear span
$\mathcal{2}$	the connected groupoid with two points and four arrows
$\mathcal{C}, \mathcal{D}$	strict monoidal category
$\mathcal{E}$	conditional expectation
$\mathcal{E}_s$	source 'right conditional expectation, page 34
$\mathcal{E}_t$	target 'left conditional expectation', page 34
$\mathcal{G}$	left unitary projective corepresentation

$\mathcal{T}_\varphi$ implementation * w.r.t. φ

$\mathfrak{A}_2$ a Tomita algebra inside N_2, page 175

$\mathfrak{G}$ locally compact group

$\mathscr{D}$ domain of a map

$\mathscr{H}, \mathscr{G}$ Hilbert space

$\mathscr{H} \otimes \mathscr{G}$ tensor product of Hilbert spaces

$\mathscr{H}_\varphi$ space of left bounded vectors

$\mathscr{I}_M$ domain dual GNS map inside M_*, page 189

$\mathscr{K}$ subspace of $\mathscr{L}^2(N) \underset{\mu}{\otimes} \mathscr{L}^2(N)$, page 178

$\mathscr{L}^1_*(M)$ domain of $(S_M)_*$, page 194

$\mathscr{L}^2(N)$ standard GNS space

$\mathscr{L}^2(N, \varphi)$ GNS space w.r.t. weight φ

$\mathscr{L}^2(Q_{ij})$ constituents $\mathscr{L}^2$-space linking von Neumann algebra, page 165

$\mathscr{M}_\varphi$ space of integrable elements

$\mathscr{M}^+_\varphi$ space of positive integrable elements

$\mathscr{N}_\varphi$ space of square integrable elements

$\mathscr{T}_{\varphi,T}$ Tomita algebra w.r.t weight and operator valued weight, page 175

$\mathscr{T}_\varphi$ Tomita algebra, page 160

μ weight on base algebra; also: see page 333

∇_N modular operator of φ_N

∇_φ modular operator

$\nabla_{\hat{N}}$ spatial derivative $\varphi_{\hat{P}}$ w.r.t. $\varphi'_{\widehat{M}}$, page 230

ν relatively invariant nsf weight on object algebra measured quantum groupoid

τ_t^M scaling group

Comod-A category of right comodules

$\mathrm{Gal}_r(\widehat{Q})$ right Galois object constructed from linking von Neumann algebraic quantum groupoid $\widehat{Q}$, page 248

Ind induction functor, page 25

LQG(N) linking von Neumann algebraic quantum groupoid constructed from the right Galois object N, page 248

Mod-A category of unital right A-modules

Res restriction functor, page 25

Θ torsor map, page 258

θ right representation

θ_N standard right GNS representation

θ_Q^i right representation linking algebra on i-th row, page 166

θ_{ij}^k right representation on k-th row of ij-th part, page 166

$\tilde{G}$ Galois isometry, page 202

ε_A counit

φ' commutant weight of φ on N', page 158

φ, ψ weights

φ^{op} opposite weight

φ_N δ_M-invariant weight on N

$\varphi_P \oplus \varphi_M$ balanced weight

φ_A left invariant functional

$\widehat{\Gamma}_M$ scaled GNS-map for $\varphi_{\widehat{M}'}$, page 191

$\widehat{\pi}'_\alpha$ restriction Galois homomorphism to $\widehat{M}'$, page 200

$\widehat{\pi}^j$ left representation $\widehat{Q}$ on j-th column standard GNS space, page 245

$\widehat{\pi}_\alpha$ left representation opposite to $\widehat{\theta}'_\alpha$, page 200

$\widehat{\pi}^j_{ik}$ left representation of $\widehat{Q}_{ik}$ on j-th column standard GNS space, page 245

$\widehat{\theta}'_\alpha$ $J_N \widehat{\pi}'_\alpha(\,\cdot\,)^* J_N$, page 200

$\widehat{\theta}_\alpha$ right representation opposite to $\widehat{\pi}'_\alpha$, page 200

$\widehat{A}$ dual algebraic quantum group

$\widehat{B}, \widehat{C}$ dual of Galois object

$\widehat{E}^u$ universal C*-algebra of the linking von Neumann algebraic quantum groupoid $\widehat{Q}$

$\widehat{M}'$ commutant of the dual

$\widehat{N}$ $\widehat{M}$-intertwiners between $\mathscr{L}^2(M)$ and $\mathscr{L}^2(N)$, page 201

$\widehat{O}$ $\widehat{M}$-intertwiners between $\mathscr{L}^2(N)$ and $\mathscr{L}^2(M)$, page 201

$\widehat{Q}$ linking von Neumann algebraic quantum groupoid (dual to Galois object)

$\widehat{W}_{ik}$ part of $W_{\widehat{Q}}$ in $\widehat{Q}_{ki} \otimes Q_{ik}$, page 245

$\widehat{W}^j_{ik}$ $\widehat{W}_{ik}$, with $\widehat{\pi}^j$ applied to first leg, page 245

$\xi\ {}_\theta\underset{\varphi}{\otimes}_\pi\ \eta$, $\xi \underset{\varphi}{\otimes} \eta$ elementary tensor in Connes-Sauvageot tensor product

${}_\varphi\mathscr{H}$ space of right bounded vectors

$A \underset{\min}{\otimes} B$ minimal tensor product of C*-algebras

A,B,C,D,E,F,L algebras

A,D ((multiplier) Hopf)(C*-)algebras, algebraic quantum group

A-Mod category of unital left A-modules

A-Comod category of left comodules

A^u, D^u universal C*-algebras

A^{cop} opposite coalgebra

A^{op} opposite algebra

B, C Morita module or equivalence bimodule

C inverse equivalence bimodule of B, opposite algebra of B

C_N conjugation from N to N', page 158

$C_{\mathscr{H}}$ canonical anti-*-isomorphism from $B(\mathscr{H})$ to $B(\overline{\mathscr{H}})$

d target map for a measured quantum groupoid

d_{M_1} restriction modular element, page 216

E algebra underlying a (co-)linking algebra, reduced C*-algebra underlying a co-linking von Neumann algebraic quantum groupoid

E^s source subalgebra, page 33

E^t target subalgebra, page 33

E^u universal C*-algebra of the co-linking von Neumann algebraic quantum groupoid Q

E_{ij} components of a (3×3) (co-)linking (weak (multiplier) Hopf) algebra

F functor, algebra of coinvariants, embedding of a quantum subgroup

f source map for a measured quantum groupoid

G, H Galois map, page 89

J_N standard modular conjugation

J_N, J_O constituents modular conjugation linking von Neumann algebra, page 166

$J_{\mathscr{H}}$ Conjugation anti-unitary from $\mathscr{H}$ to $\overline{\mathscr{H}}$

J_φ modular conjugation

k a field

L object algebra, page 34 and 317

$L^{\theta,\varphi}(\xi)$,L_ξ left multiplication with left bounded vector

l_ξ, r_ξ left/right ‘creation operators’, page 13

l_a left multiplication with a

$M \otimes N$ spatial tensor product of von Neumann algebras

$M(A)$ multiplier algebra

$M(B), M(C)$ multiplier envelopes, page 79

M, N, O, P, Q, Y von Neumann algebras

M^{cop} co-opposite von Neumann algebraic quantum group

$M_2(A)$ two-by-two matrices over the algebra A

M_A multiplication map of an algebra, page 17

m_V module structure, page 19

$M_{1,2}(A \odot B), \ldots$ restricted multiplier algebras

N' commutant von Neumann algebra

$N, (N, \alpha_N)$ (right) Galois object

$N \underset{\alpha}{\rtimes} M, N \rtimes M$ crossed product von Neumann algebra

N^+ positive cone

$N^{+,\text{ext}}$ extended positive cone

N^α von Neumann algebra of coinvariants

N^{op} opposite von Neumann algebra

N_*^+ positive cone predual

$N_0 \subseteq N \subseteq N_2 \subseteq N_3 \subseteq \ldots$ tower construction

$N_1 \underset{L}{{}_s*_t} N_2$ fibre product of von Neumann algebras

p support projection Galois homomorphism, page 206

P_N scaling operator of N, page 223

$P_{\varphi_N}^{it}$ page 209

P_M^{it} unitary implementation scaling group

W_M left regular corepresentation

W_Q left regular corepresentation co-linking von Neumann algebraic quantum groupoid

$W_{\hat{Q}}$ left regular corepresentation linking von Neumann algebraic quantum groupoid

W_{ik}^j $\Sigma(\widehat{W}_{ik}^j)^*\Sigma$, page 247

$x \underset{\varphi}{\otimes} y$, $x \underset{N}{\otimes} y$ balanced tensor product of operators

Index