# A note on the von Neumann algebra underlying some universal compact quantum groups


Kenny De Commer*

Department of Mathematics, K.U. Leuven

Celestijnenlaan 200B, 3001 Leuven, Belgium

e-mail: kenny.decommer@wis.kuleuven.be



**Abstract**

We show that for $F \in GL(2, \mathbb{C})$, the von Neumann algebra associated to the universal quantum group $A_u(F)$ is a free Araki-Woods factor.


## Introduction

It is a classical theorem that any compact Lie group is a closed subgroup of some $U(n)$. In [5], a class of quantum groups was introduced which plays the same rôle with respect to the compact matrix quantum groups (introduced in [8], but there called compact quantum *pseudo*groups). These universal quantum groups were denoted $A_u(F)$, where the parameter $F$ takes values in invertible matrices over $\mathbb{C}$. In [1], the representation theory of the $A_u(F)$ was investigated, and it was shown that the irreducible representations are naturally labeled by the free monoid with two generators. Also on the level of the 'function algebra' of $A_u(F)$, freeness manifests itself: it was shown in [1] that the (normalized) trace of the fundamental representation is a circular element w.r.t. the Haar state (in the sense of Voiculescu, see [6]). Furthermore, the von Neumann algebra associated to $A_u(I_2)$, where $I_2$ is the unit matrix in $GL(2, \mathbb{C})$, is actually isomorphic to the free group factor $\mathscr{L}(\mathbb{F}_2)$.

In this note, we generalize this last result by showing that for $0 < q \leq 1$, the von Neumann algebra underlying the universal quantum group $A_u(F)$ with $F = \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}$ is a free Araki-Woods factor ([4]), namely the one associated to the orthogonal representation

$$t \rightarrow \begin{pmatrix} \cos(t \ln q^2) & -\sin(t \ln q^2) \\ \sin(t \ln q^2) & \cos(t \ln q^2) \end{pmatrix}$$

of $\mathbb{R}$ on $\mathbb{R}^2$. The proof of this fact uses a technique similar to the one of Banica for the case $F = I_2$, combined with results from [3] (which are based on the matrix model techniques from [4]). Since

$$A_u(F) = A_u(\lambda U|F|U^*)$$


*Research Assistant of the Research Foundation - Flanders (FWO - Vlaanderen).

for any $\lambda \in \mathbb{R}_0^+$ and any unitary $U$ (see [1]), we obtain that all $A_u(F)$ with $F \in GL(2,\mathbb{C})$ have free Araki-Woods factors as their associated von Neumann algebras.

*Remarks on notation:* We denote by $\odot$ the algebraic tensor product of vector spaces over $\mathbb{C}$, and by $\otimes$ the spatial tensor product between von Neumann algebras or Hilbert spaces. If $M$ is a von Neumann algebra and $x_1, x_2, \ldots$ are elements in $M$, we denote by $W^*(x_1, x_2, \ldots)$ the von Neumann subalgebra of $M$ which is the $\sigma$-weak closure of the unital $^*$-algebra generated by the $x_i$.

# 1 Preliminaries

In this preliminary section, we will give, for the sake of economy, ad hoc definitions of the von Neumann algebras associated to the $A_u(F)$ and $A_o(F)$ quantum groups ([5]), and of the free Araki-Woods factors ([4]), for special values of their parameters.

*Throughout this section, we fix a number $0 < q < 1$.*

**Definition 1.1.** *We define the C*-algebra $C_u(H)$ as the universal enveloping C*-algebra of the unital $^*$-algebra generated by elements $a$ and $b$, with defining relations*

$$\begin{cases} a^*a + b^*b = 1 & ab = qba \\ aa^* + q^2 bb^* = 1 & a^*b = q^{-1}ba^* \\ & bb^* = b^*b. \end{cases}$$

*Remark:* $C_u(H)$ is the (universal) C*-algebra associated with the quantum group $H = SU_q(2)$. In [1], Proposition 5, it is shown that this equals the quantum group $A_o(\begin{pmatrix} 0 & 1 \\ -q^{-1} & 0 \end{pmatrix})$.

The following fact is found in [9].

**Lemma 1.2.** *Let $\mathscr{H}$ be the Hilbert space $l^2(\mathbb{N}) \otimes l^2(\mathbb{Z})$, whose canonical basis elements we denote as $\xi_{n,k}$ (and with the convention $\xi_{n,k} = 0$ when $n < 0$). Then there exists a faithful unital $^*$-representation of $C_u(H)$ on $\mathscr{H}$, determined by*

$$\begin{cases} \pi(a)\,\xi_{n,k} = \sqrt{1-q^{2n}}\xi_{n-1,k}, \\ \pi(b)\xi_{n,k} = q^n\,\xi_{n,k+1}. \end{cases}$$

**Definition 1.3.** *In the notation of the previous lemma, denote by $\psi$ the state*

$$\psi(x) = (1-q^2)\sum_{n\in\mathbb{N}} q^{2n}\langle \pi(x)\xi_{n,0}, \xi_{n,0}\rangle$$

*on $C_u(H)$. Then $\psi$ is called the* Haar state *on $C_u(H)$.*

Of course, this name is motivated by the further compact quantum group structure on $C_u(H)$, which we will however not need in the following.

**Definition 1.4.** *The von Neumann algebra $\mathscr{L}^\infty(H)$ is defined to be the $\sigma$-weak closure of $C_u(H)$ in its GNS-representation with respect to the Haar state $\psi$.*

We then continue to write $\psi$ for the extension of $\psi$ to a normal state on $\mathscr{L}^\infty(H)$.

**Notation 1.5.** *We will further use the following notations:*

- *The matrix units of $B(l^2(\mathbb{N}))$ w.r.t. the canonical basis of $l^2(\mathbb{N})$ are written $e_{ij}$.*
- *We denote $\omega$ for the normal state $\omega(e_{ij}) = \delta_{i,j}(1-q^2)q^{2i}$ on $B(l^2(\mathbb{N}))$.*
- *We denote by $S \subseteq \mathscr{L}(\mathbb{Z})$ the shift operator $\xi_k \to \xi_{k+1}$ on $l^2(\mathbb{Z})$.*
- *We denote by $\tau$ the state on $\mathscr{L}(\mathbb{Z})$ which makes $S$ into a Haar unitary with respect to $\tau$.*

This last fact simply means that $\tau(S^n) = 0$ for $n \in \mathbb{Z}_0$.

We will use the terminology 'W*-probability space' when talking about a von Neumann algebra with some fixed normal state on it. An isomorphism between two W*-probability spaces is then a *-isomorphism between the underlying von Neumann algebras, preserving the associated fixed states.

**Lemma 1.6.** *There is a natural isomorphism*

$$(\mathscr{L}^\infty(H), \psi) \to (B(l^2(\mathbb{N})) \otimes \mathscr{L}(\mathbb{Z}), \omega \otimes \tau)$$

*of W*-probability spaces.*

*Proof.* By the construction of $\psi$, we may identify $\mathscr{L}^\infty(H)$ with $\pi(C_u(H))''$, and it is then sufficient to prove that this last von Neumann algebra equals $B(l^2(\mathbb{N})) \otimes \mathscr{L}(\mathbb{Z})$. Clearly, $\pi(C_u(H))'' \subseteq B(l^2(\mathbb{N})) \otimes \mathscr{L}(\mathbb{Z})$. By functional calculus on $a$ and $b$, we have $e_{ij} \otimes S^n \in \pi(C_u(H))''$ for all $i, j \in \mathbb{N}$ and $n \in \mathbb{Z}$, so in fact equality holds. □

We will always write $(1 \otimes S)$ for the copy of $S \in \mathscr{L}(\mathbb{Z})$ inside $\mathscr{L}^\infty(H)$. Hence there should be no notational confusion in the following definition.

**Definition 1.7.** *The W*-probability space $(\mathscr{L}^\infty(G), \varphi)$ is defined as*

$$(W^*(Sa, Sb, Sa^*, Sb^*), (\tau * \psi)_{|\mathscr{L}^\infty(G)}) \subseteq (\mathscr{L}(\mathbb{Z}), \tau) * (\mathscr{L}^\infty(H), \psi).$$

*Remark:* By [1], Théorème 1.(iv), the von Neumann algebra $\mathscr{L}^\infty(G)$ will coincide with the von Neumann algebra associated with the universal quantum group $A_u(\begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix})$, and $\varphi$ with its Haar state.

Recall that the state $\omega$ was introduced in Notation 1.5.

**Definition 1.8.** *([4], Corollary 4.9) By a* free Araki-Woods factor (at parameter $q^2$)*, we mean a W*-probability space $(N, \phi)$ isomorphic to the free product $(\mathscr{L}(\mathbb{Z}), \tau) * (B(l^2(\mathbb{N})), \omega)$.*

# 2 $\mathscr{L}^\infty(G)$ is free Araki-Woods

*Throughout this section, we again fix a number $0 < q < 1$. We also continue to use the notations introduced in the previous section.*

We proceed to prove the following theorem.

**Theorem 2.1.** *The W*-probability space $(\mathscr{L}^\infty(G), \varphi)$ is a free Araki-Woods factor at parameter $q^2$.*

By the remark after Definition 1.7 and the remarks in the introduction, this will imply that if $F \in GL(2, \mathbb{C})$, then the von Neumann algebra associated to $A_u(F)$ is the free Araki-Woods factor at parameter $\frac{\lambda_1}{\lambda_2}$, where $\lambda_1 \leq \lambda_2$ are the eigenvalues of $F^*F$ (where we take $\mathscr{L}(\mathbb{F}_2)$ to be the free

Araki-Woods factor at parameter 1).

The proof of Theorem 2.1 will be preceded by three lemmas. Consider the following von Neumann subalgebras of $(\mathscr{L}(\mathbb{Z}),\tau) * (\mathscr{L}^\infty(H),\psi)$:

$$(M_1,\varphi_1) = (W^*(S(1\otimes S)), (\tau*\psi)_{|M_1})$$

and

$$(M_2,\varphi_2) = (W^*((1\otimes S^*)a, (1\otimes S^*)b, (1\otimes S^*)a^*, (1\otimes S^*)b^*), (\tau*\psi)_{|M_2}).$$

**Lemma 2.2.** *The von Neumann algebras $M_1$ and $M_2$ are free with respect to each other, and $\mathscr{L}^\infty(G)$ is the smallest von Neumann subalgebra of $\mathscr{L}(\mathbb{Z}) * \mathscr{L}^\infty(H)$ which contains them.*

*Proof.* The proof is entirely similar to the one of Théorème 6 in [1]. First of all, remark that $S(1\otimes S)$ is the unitary part in the polar decomposition of $Sb$, so that $S(1\otimes S)$ is in $\mathscr{L}^\infty(G)$. Then of course

$$(1\otimes S^*)a = (1\otimes S^*)S^*\cdot Sa$$

is in $\mathscr{L}^\infty(G)$, and similarly for the other generators of $M_2$. Hence $M_1$ and $M_2$ indeed generate $\mathscr{L}^\infty(G)$.

The proof of the freeness of $M_1$ w.r.t. $M_2$ is based on a small alteration of Lemme 8 of [1].

**Lemma.** *Let $(A,\phi)$ be a unital $^*$-algebra together with a functional $\phi$ on it. Let $B\subseteq A$ be a unital sub-$^*$-algebra, and $d\in B$ a unitary in the center of $B$ such that $\phi(d)=\phi(d^*)=0$. Let $u\in A$ be a Haar unitary which is $^*$-free from $B$ w.r.t. $\phi$. Then $ud$ is a Haar unitary which is $^*$-free from $B$ w.r.t. $\phi$.*

*Proof.* This is precisely Lemme 8 of [1], with the condition '$\phi$ is a trace' replaced by '$d$ is in the center of $B$'. However, the proof of that lemma still applies ad verbam. ☐

We can then apply this lemma to get that $S(1\otimes S)$ is $^*$-free w.r.t. $\mathscr{L}^\infty(H)$, by taking $(A,\phi) = (\mathscr{L}(\mathbb{Z}),\tau) * (\mathscr{L}^\infty(H),\psi)$, $B=\mathscr{L}^\infty(H)$, $d = 1\otimes S$ and $u=S$. *A fortiori*, we will then have $M_1$ free w.r.t. $M_2$. ☐

**Lemma 2.3.** *We have*

$$(M_1,\varphi_1)\cong(\mathscr{L}(\mathbb{Z}),\tau)$$

*and*

$$(M_2,\varphi_2)\cong(B(l^2(\mathbb{N}))\otimes\mathscr{L}(\mathbb{Z}),\omega\otimes\tau).$$

*Proof.* The fact that $(M_1,\varphi_1)\cong(\mathscr{L}(\mathbb{Z}),\tau)$ is of course trivial. We want to show that $(M_2,\varphi_2)\cong(B(l^2(\mathbb{N}))\otimes\mathscr{L}(\mathbb{Z}),\omega\otimes\tau)$.

We have that $1\otimes S^2$ is in $M_2$, since this is the adjoint of the unitary part of the polar decomposition of $(1\otimes S^*)b^*$. Also all $e_{ii}\otimes 1$ are in $M_2$, by functional calculus on the positive part of this polar decomposition. Hence, by multiplying $(1\otimes S^*)a$ or $(1\otimes S^*)a^*$ to the left with the $e_{ii}\otimes 1$, and possibly multiplying with $1\otimes S^2$, we conclude that the $e_{ij}\otimes S^{i-j}$ with $|i-j|=1$ are in $M_2$. But then also all $f_{ij}=e_{ij}\otimes S^{i-j}$ with $i,j\in\mathbb{N}$ are in $M_2$, and it is not hard to see that in fact $M_2 = W^*(f_{ij},(1\otimes S^2))$. Since $\psi(f_{ij}(1\otimes S^2)^n) = (\omega\otimes\tau)(e_{ij}\otimes S^n)$ by an easy calculation, we are done. ☐

**Lemma 2.4.** *The $W^*$-probability space $(N,\phi) := (\mathscr{L}(\mathbb{Z}),\tau) * (B(l^2(\mathbb{N}))\otimes\mathscr{L}(\mathbb{Z}),\omega\otimes\tau)$ is a free Araki-Woods factor at parameter $q^2$.*

*Proof.* The proof is completely similar to the one of Theorem 3.1 of [3]. Denote $(N,\theta) = (\mathscr{L}(\mathbb{Z}),\tau) * (B(l^2(\mathbb{N})),\omega)$, and denote $\phi_0 = \frac{1}{1-q^2}\phi$ and $\theta_0 = \frac{1}{1-q^2}\theta$. Then by Proposition 3.10 of [3], we will have that

$$(e_{00}Me_{00},\phi_0) \cong (\mathscr{L}(\mathbb{Z}),\tau) * (e_{00}Ne_{00},\theta_0).$$

By Proposition 2.7 in [3] (which is based on the proof of Theorem 5.4 and Proposition 6.3 in [4]) and the remark before it, we know that $(e_{00}Ne_{00},\theta_0)$ as well as $(N,\theta) \cong (e_{00}Ne_{00},\theta_0) \otimes (B(l^2(\mathbb{N})),\omega)$ are free Araki-Woods factors at parameter $q^2$. By the free absorption property ([4], Corollary 5.5), $(e_{00}Me_{00},\phi_0)$ is a free Araki-Woods factor at parameter $q^2$, and hence also $(M,\phi) \cong (e_{00}Me_{00},\phi_0) \otimes (B(l^2(\mathbb{N})),\omega)$ is.

□

*Proof (of Theorem 2.1).* By the first two lemmas, $(\mathscr{L}^\infty(G),\varphi)$ is isomorphic to the free product of $(\mathscr{L}(\mathbb{Z}),\tau)$ with $(B(l^2(\mathbb{N})) \otimes \mathscr{L}(\mathbb{Z}),\omega\otimes\tau)$, which by the third lemma is a free Araki-Woods factor at parameter $q^2$. □

**Acknowledgement:** The motivation for this paper comes from a question posed by Stefaan Vaes concerning the validity of Theorem 2.1.

# References


[1] T. Banica, Le groupe quantique compact libre $U(n)$, *Comm. Math. Phys.* **190** (1997), 143–172.

[2] L. Barnett, Free product von Neumann algebras of type III, *Proc. Amer. Math. Soc.* **123** (1995), 543–553.

[3] C. Houdayer, On some free products of von Neumann algebras which are free Araki-Woods factors, *Int. Math. Res. Notices 2007* (2007), article ID rnm098, 21 pages.

[4] D. Shlyakhtenko, Free quasi-free states, *Pacific Journal of Mathematics* **177** (2) (1997), 329–368.

[5] A. Van Daele and S.Z. Wang, Universal quantum groups, *Int. J. Math.* **7** (2) (1996), 255–264.

[6] D.V. Voiculescu, Circular and semicircular systems and free product factors, *Operator Algebras, Unitary Representations, Enveloping Algebras and Invariant Theory, Progress in Mathematics* **92**, Birkhäuser, Boston (1990).

[7] D.V. Voiculescu, K.J. Dykema and A. Nica, Free random variables, *CRM Monograph Series* **1**, AMS, Providence (1992).

[8] S.L. Woronowicz, Compact matrix pseudogroups, *Comm. Math. Phys.* **111** (1987), 613–665.

[9] S.L. Woronowicz, Twisted $SU(2)$ group. An example of a non-commutative differential calculus, *Publ. RIMS, Kyoto University* **23** (1987), 117–181.