FREE ACTIONS OF COMPACT QUANTUM GROUPS ON UNITAL C*-ALGEBRAS

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Abstract. Let F be a field, Γ a finite group, and $\mathrm{Map}(\Gamma, F)$ the Hopf algebra of all set-theoretic maps $\Gamma \to F$. If E is a finite field extension of F and Γ is its Galois group, the extension is Galois if and only if the canonical map $E \otimes_F E \to E \otimes_F \mathrm{Map}(\Gamma, F)$ resulting from viewing E as a $\mathrm{Map}(\Gamma, F)$-comodule is an isomorphism. Similarly, a finite covering space is regular if and only if the analogous canonical map is an isomorphism. In this paper we extend this point of view to actions of compact quantum groups on unital C^*-algebras. We prove that such an action is free if and only if the canonical map (obtained using the underlying Hopf algebra of the compact quantum group) is an isomorphism. In particular, we are able to express the freeness of a compact Hausdorff topological group action on a compact Hausdorff topological space in algebraic terms.

Contents

Introduction

A *compact quantum group* [W-SL87, W-SL98] is a unital C^*-algebra H with a given unital injective $*$-homorphism Δ (referred to as comultiplication)

$$\Delta\colon H \longrightarrow H \underset{\min}{\otimes} H \tag{0.1}$$

which is coassociative i.e. there is commutativity in the diagram

$$\begin{array}{ccc} H & \xrightarrow{\Delta} & H \underset{\min}{\otimes} H \\ \downarrow{\scriptstyle\Delta} & & \downarrow{\scriptstyle\Delta\otimes\mathrm{id}} \\ H \underset{\min}{\otimes} H & \xrightarrow[\mathrm{id}\otimes\Delta]{} & H \underset{\min}{\otimes} H \underset{\min}{\otimes} H \end{array} \tag{0.2}$$

such that the two-sided cancellation property holds:

$$\{(a \otimes 1)\Delta(b) \mid a, b \in H\}^{\mathrm{cls}} = H \underset{\min}{\otimes} H = \{\Delta(a)(1 \otimes b) \mid a, b \in H\}^{\mathrm{cls}}. \tag{0.3}$$

Here $\otimes_{\min}$ denotes the spatial tensor product of C^*-algebras and cls denotes the closed linear span of a subset of a Banach space.

Let A be a unital C^*-algebra and $\delta : A \to A \otimes_{\min} H$ an injective unital $*$-homomorphism. We call δ a *coaction* of H on A (or an action of the compact quantum group (H, Δ) on A) if

(1) $(\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \Delta) \circ \delta$ (coassociativity),
(2) $\{\delta(a)(1 \otimes h) \mid a \in A,\, h \in H\}^{\mathrm{cls}} = A \underset{\min}{\otimes} H$ (counitality).

By definition [E-DA00], the coaction δ is *free* if and only if

$$\{(x \otimes 1)\delta(y) \mid x, y \in A\}^{\mathrm{cls}} = A \underset{\min}{\otimes} H. \tag{0.4}$$

Given a compact quantum group (H, Δ), we denote by $\mathcal{O}(H)$ its dense Hopf $*$-subalgebra spanned by the matrix coefficients of irreducible unitary corepresentations [W-SL98, MV98]. This is Woronowicz's Peter-Weyl theory in the case of compact quantum groups. Moreover, denoting by $\otimes$ the purely algebraic tensor product over the field $\mathbb{C}$ of complex numbers, we define the *Peter-Weyl subalgebra* of A (cf. [P-P95]) as

$$\mathcal{P}_H(A) := \{\, a \in A \,|\, \delta(a) \in A \otimes \mathcal{O}(H) \,\}. \tag{0.5}$$

Using the coassociativity of δ, one can check that $\mathcal{P}_H(A)$ is a right $\mathcal{O}(H)$-comodule algebra. In particular, $\mathcal{P}_H(H) = \mathcal{O}(H)$. The assignment $A \mapsto \mathcal{P}_H(A)$ is functorial with respect to equivariant unital *-homomorphisms and comodule algebra maps. We call it the *Peter-Weyl functor.*

The theorem of this paper is:

Theorem 0.1. *Let A be a unital C^*-algebra equipped with an action of a compact quantum group (H, Δ) given by $\delta\colon A \to A \otimes_{\min} H$. Denote by $B = A^{\mathrm{co}H} := \{a \in A \mid \delta(a) = a \otimes 1\}$ the unital C^*-subalgebra of coaction-invariants. The action is free if and only if the canonical map*

$$can\colon \mathcal{P}_H(A) \underset{B}{\otimes} \mathcal{P}_H(A) \longrightarrow \mathcal{P}_H(A) \otimes \mathcal{O}(H)$$
$$can\colon x \otimes y \longmapsto (x \otimes 1)\delta(y)$$

is bijective. (Here the tensor product over an algebra denotes the purely algebraic tensor product over that algebra.)

Our result generalizes Woronowicz's Peter-Weyl theory from compact quantum groups to compact quantum principal bundles. In the spirit of the Woronowicz theorem, our result replaces the original functional analysis formulation of free action with a much more algebraic condition.

We now proceed to explaining the topological meaning of our main result. The classical setting not only allows one to develop an intuition, but also is an immediate application of the main theorem. However, this part is not necessary for following its proofs — one can pass from here directly to Section 3.

Let G be a compact Hausdorff topological group acting (by continuous right action) on a compact Hausdorff topological space X

$$X \times G \longrightarrow X. \tag{0.6}$$

It is immediate that the action is free i.e. $xg = x \Longrightarrow g = e$ (where e is the identity element of G) if and only

$$\begin{aligned} X \times G &\longrightarrow X \underset{X/G}{\times} X \\ (x, g) &\longmapsto (x, xg) \end{aligned} \tag{0.7}$$

is a homeomorphism. Here $X \times_{X/G} X$ is the subset of $X \times X$ consisting of pairs (x_1, x_2) such that x_1 and x_2 are in the same G-orbit.

This is equivalent to the assertion that the $*$-homomorphism

$$C(X \underset{X/G}{\times} X) \longrightarrow C(X \times G) \tag{0.8}$$

obtained from the above map $(x, g) \mapsto (x, xg)$ is an isomorphism. Here, as usual, $C(Y)$ denotes the C^*-algebra of all continuous complex-valued functions on the compact Hausdorff space Y.

In turn, this assertion is readily proved equivalent to

$$\{(x \otimes 1)\delta(y) \mid x, y \in C(X)\}^{\mathrm{cls}} = C(X) \underset{\min}{\otimes} C(G), \tag{0.9}$$

where

$$\delta\colon C(X) \longrightarrow C(X) \underset{\min}{\otimes} C(G) \tag{0.10}$$

is the $*$-homomorphism obtained from the map $X \times G \to X$ via the formula $(\delta(f)(g))(x) = f(xg)$. Hence in the case of a compact group acting on a compact space "free action" agrees with "free action" as defined in the setting of a compact quantum group acting on a unital C^*-algebra. Thus we can formulate the commutative case of Theorem 0.1 as follows.

Theorem 0.2. *Let G be a compact Hausdorff group acting continuously on a compact Hausdorff space X. The action is free if and only if the canonical map*

$$can\colon \mathcal{P}_{C(G)}(C(X)) \underset{C(X/G)}{\otimes} \mathcal{P}_{C(G)}(C(X)) \longrightarrow \mathcal{P}_{C(G)}(C(X)) \otimes \mathcal{O}(C(G)) \tag{0.11}$$

is an isomorphism.

Observe that even in the above special case of a compact group acting on a compact space, a proof is required for the equivalence of "free action" and the bijectivity of the canonical map (Galois condition). This theorem brings an essential new algebraic tool (strong connection) to the realm of compact principal bundles. The Peter-Weyl algebra in this setting becomes a very natural object, notably the algebra of continuous global sections of an associated $\mathcal{O}(C(G))$-fibre bundle, where $\mathcal{O}(C(G))$ is a topological vector space for the direct limit topology and the multiplication of sections is pointwise. Although Theorem 0.2 is a special case of Theorem 0.1, its proofs are not special cases of the proof of Theorem 0.1. Therefore we treat Theorem 0.2 separately.

In the first two sections we provide proofs of Theorem 0.2. The first proof has no noncommutative counterpart as it relies on the local triviality of principal bundles with compact Lie structure groups. The second proof is global in nature. It uses the strong monoidality of the Serre-Swan Theorem, which is later reflected in the noncommutative setting of Theorem 4.3. The third section proves the main result (Theorem 0.1) by taking advantage of an underlying Hilbert module structure. Then we explore the general algebraic setting of principal coactions in Section 4. It becomes the common denominator for free actions of compact Hausdorff groups on compact Hausdorff spaces and principal actions of affine algebraic groups on affine schemes. We end with an appendix devoted to the equivalence of the regularity of a finite covering and the bijectivity of the canonical map (0.11).

1. Approximation proof

To be consistent with general notation, we should only use C^*-algebras $C(G)$, $C(X)$, etc., rather than spaces themselves. However, this would make formulas too cluttered, so that throughout this section we consistently omit writing $C(\ )$ in the subscript and the argument of the Peter-Weyl functor.

The proof consists of the following six steps.

(1) Approximation by Lie-group principal bundles with the same base.
(2) Local triviality of Lie-group principal bundles (A. M. Gleason).
(3) Piecewise triviality due to the compactness of the base space.
(4) $\mathcal{P}_G(M \times G) = C(M) \otimes \mathcal{O}(G)$.
(5) $\mathcal{P}_H(A \times_B C) = \mathcal{P}_H(A) \times_{\mathcal{P}_H(B)} \mathcal{P}_H(C)$.
(6) Application of the Pullback Theorem and induction.

2. Serre-Swan proof

The main advantage of this proof over the previous one is that it does not rely on an approximation argument which works only in the classical setting. Instead, we rely on the following fundamental facts:

Theorem 2.1 ([S-R62])**.** *Let M be a compact Hausdorff topological space. Then a $C(M)$-module is finitely generated and projective if and only if it is isomorphic to the module of continuous global sections of a vector bundle over M.*

Corollary 2.2. *Let M be a compact Hausdorff topological space. Then $\Gamma\colon$ is a functor giving an equivalence of categories. Moreover, with respect to the natural map $\beta\colon$ Γ is strongly monoidal.*

With the help of Corollary 2.2, the proof boils down to the following calculation:

(2.1)
$$\begin{array}{ccc} C(X) & \overset{E}{\underset{F}{\rightleftarrows}} & C_G(X, C(G)) \\ \uparrow \subseteq & & \uparrow \subseteq \\ \mathcal{P}_G(X) & \overset{E}{\underset{F}{\rightleftarrows}} & C_G^{\text{f.d.}}(X, \mathcal{O}(G)), \end{array}$$

(2.2)
$$E(f)(x)(g) := f(xg), \quad F(\alpha)(x) := \alpha(x)(e), \quad E \circ F = \text{id}, \quad F \circ E = \text{id}.$$

$$\mathcal{P}_G(X) \underset{C(X/G)}{\overset{\text{alg}}{\otimes}} \mathcal{P}_G(X) \overset{E \otimes E}{\longrightarrow} C_G^{\text{f.d.}}(X, \mathcal{O}(G)) \underset{C(X/G)}{\overset{\text{alg}}{\otimes}} C_G^{\text{f.d.}}(X, \mathcal{O}(G)) \overset{\text{diag}}{\longrightarrow}$$
$$C_G^{\text{f.d.}}(X, \mathcal{O}(G) \underset{\text{alg}}{\otimes} \mathcal{O}(G)) \overset{W^* \circ}{\longrightarrow} C_{G,\text{id}}^{\text{f.d.}}(X, \mathcal{O}(G) \underset{\text{alg}}{\otimes} \mathcal{O}(G)) \overset{\sum_i (\text{id} \otimes e^i) \otimes e_i}{\longrightarrow}$$
$$C_G^{\text{f.d.}}(X, \mathcal{O}(G)) \underset{\text{alg}}{\otimes} \mathcal{O}(G) \overset{F \otimes \text{id}}{\longrightarrow} \mathcal{P}_G(X) \underset{\text{alg}}{\otimes} \mathcal{O}(G), \text{ where } W(g, g') := (g, gg').$$

3. General proof

One direction of the equivalence in Theorem 0.1 is immediate. That is, if $\mathcal{P}_H(A)$ is principal, then δ is free. Indeed, by the Galois condition (1) in the definition of principality, we have

(3.3)
$$(\mathcal{P}_H(A) \otimes \mathbb{C})\delta(\mathcal{P}_H(A)) = \mathcal{P}_H(A) \otimes \mathcal{O}(H).$$

As the right hand side is a dense subspace of $A \underset{\min}{\otimes} H$ [P-P95, Theorem 1.5.1], we obtain the density condition defining freeness.

For the converse part, we need some preparations. If (V, δ_V) is a finite-dimensional comodule of H, we write H_V for the smallest subspace of H such that $\delta_V(V) \subseteq V \otimes H_V$. We write $A_V = \{a \in A \mid \delta(a) \in A \otimes H_V\}$, which is consistent with the above notation in the case $(A, \delta) = (H, \Delta)$.

One can define a continuous projection map E_V from A onto A_V as follows [P-P95, Theorem 1.5.1]. Let us call two finite-dimensional comodules of H *disjoint* if the set of morphisms between them only contains the zero map. Then E_V is the unique endomorphism of A which is the identity on A_V and which vanishes on A_W for W any finite-dimensional comodule disjoint from V. The same notation will be used in the case of H itself. The following important equivariance property is easily proven:

$$\delta \circ E_V = (\mathrm{id} \otimes E_V) \circ \delta. \tag{3.4}$$

When V is the trivial representation, we write the projection map as $E_B : A \to B$, where B is the algebra of coaction-invariants.

We will need the following pivotal result.

Lemma 3.1 ([DC-Y12])**.** *Let (A, δ) be a free coaction for (H, Δ), and let V be a finite-dimensional comodule for (H, Δ). Then A_V is finitely generated projective as a right B-module* [DC-Y12, Theorem 1.2]. *Moreover, $\|a\|_B = \|\langle a, a\rangle_B\|^{1/2}$ defines a norm on A_V with respect to which it is complete, and the $\|\cdot\|_B$-norm is equivalent with the C^*-norm on A_V* [DC-Y12, Corollary 2.6].

Remark 3.2.

- The lemma shows that A_V equipped with the B-valued inner product $\langle a, b\rangle_B = E_B(a^*b)$ becomes a right Hilbert B-module [L-EC95].
- In case $H = C(G)$ for G a compact group, the fact that A_V is finitely generated projective is well-known, as A_V can be realized as the space of sections of an associated vector bundle.

We will need the following lemma concerning the interior tensor product of Hilbert modules [L-EC95, Chapter 4].

Lemma 3.3. *Let C and D be unital C^*-algebras. Let $(\mathscr{E}, \langle\,\cdot\,,\,\cdot\,\rangle_{\mathscr{E}})$ be a right Hilbert C-module which is finitely generated projective as a right C-module. Let $(\mathscr{F}, \langle\,\cdot\,,\,\cdot\,\rangle_{\mathscr{F}})$ be an arbitrary right Hilbert D-module, and $\pi : C \to \mathcal{L}(\mathscr{F})$ a unital *-homorphism of C into the C^*-algebra of adjointable operators on $\mathscr{F}$. Then the algebraic tensor product $\mathscr{E} \overset{alg}{\underset{C}{\otimes}} \mathscr{F}$ is a right Hilbert D-module with respect to the inner product*

$$\langle x \otimes y, z \otimes w\rangle = \langle y, \pi(\langle x, z\rangle_C)w\rangle_D. \tag{3.5}$$

Proof. We are to show that the semi-norm $\|z\| = \|\langle z, z\rangle_D\|^{1/2}$ on $\mathscr{E} \overset{\mathrm{alg}}{\underset{C}{\otimes}} \mathscr{F}$ is in fact a norm with respect to which the space is complete. The statement obviously holds with $\mathscr{E} = C^n$, the n-fold direct sum of the standard right C-module C. In general, the assumptions on $\mathscr{E}$ guarantee that it can be realized as a direct summand of C^n, so that the conclusion also applies for this case. ☐

Lemma 3.4. *Let (A, δ) be a free coaction for (H, Δ). Then can is surjective.*

Proof. By the freeness assumption, the image of *can* is dense in $A \otimes H$. In particular, for a given finite-dimensional comodule V and any $h \in H_V$, we can find a sequence $k_n \in \mathbb{N}$ and elements $p_{n,i}$ and $q_{n,i}$ in $\mathcal{P}_H(A)$ with $1 \leq i \leq k_n$ such that

$$\sum_{i=1}^{k_n} (p_{n,i} \otimes 1)\delta(q_{n,i}) \underset{n\to\infty}{\longrightarrow} 1 \otimes h \tag{3.6}$$

in the C*-norm. Applying $\mathrm{id}\otimes E_V$ to this expression, we see that we may take $q_{n,i}\in A_V$.

Applying δ to the first leg of (3.6), and using that $H_V\subseteq \mathcal{O}(H)$ is finite dimensional, we find that

$$\sum_{i=1}^{k_n}(\delta(p_{n,i})\otimes 1)(\mathrm{id}\otimes(\mathrm{id}\otimes S)\Delta)\delta(q_{n,i}) \underset{n\to\infty}{\longrightarrow} 1\otimes 1\otimes S(h), \tag{3.7}$$

where S is the antipode of $\mathcal{O}(H)$. Again by the finite dimensionality of H_V, multiplying the second and third legs is a continuous operation, so that

$$\sum_{i=1}^{k_n}\delta(p_{n,i})(q_{n,i}\otimes 1) \underset{n\to\infty}{\longrightarrow} 1\otimes S(h). \tag{3.8}$$

Note that $S(h)\in H_{\bar V}$, where $\bar V$ is the contragredient of V. Applying $\mathrm{id}\otimes E_{\bar V}$ to the above limit, we see that we can realize (3.8) with $p_{n,i}\in A_{\bar V}$ and $q_{n,i}\in A_V$.

Consider now

$$G_V : A_{\bar V}\overset{\mathrm{alg}}{\underset{B}{\otimes}} A_V \to A_{\bar V\otimes V}\otimes H_V, \qquad a\otimes b\mapsto \delta(a)(1\otimes b). \tag{3.9}$$

The left hand side becomes an interior tensor product of right Hilbert B-modules, by Lemma 3.3. On the other hand, equipping H_V with its standard Hilbert space structure $\langle h,k\rangle = \varphi_H(h^*k)$ coming from the invariant state φ_H on H, also the right hand side is a right Hilbert B-module. It is easily seen that G_V is an isometry between these Hilbert modules. Hence the range of θ_V is closed.

From (3.8) and the equivalence of C*- and Hilbert C*-module norms in Theorem 3.1, it follows that the range of G_V contains $1\otimes S(h)$. Hence we can find a finite set of elements $p_i,q_i\in\mathcal{P}_H(A)$ such that

$$\sum_i \delta(p_i)(1\otimes q_i) = 1\otimes S(h). \tag{3.10}$$

By retracing the argument at the beginning of the proof, we have that also

$$\sum_i (p_i\otimes 1)\delta(q_i) = 1\otimes h. \tag{3.11}$$

As h was arbitrary in $\mathcal{O}(H)$, it follows that *can* is surjective. □

Proof (of Theorem 0.1). As $\mathcal{O}(H)$ is co-semisimple, principality of $\mathcal{P}_H(A)$ is equivalent with the surjectivity of *can* [S-HJ90, Remark 3.9]. Theorem 0.1 thus follows from Lemma 3.4. □

Remark 3.5. Alternatively, one could adapt more directly the techniques of [DC-Y12, Theorem 3.3] to give a proof of Theorem 0.1, but the above argument has the benefit that it is based on properties which are more directly available in the setting of classical compact group actions.

4. Principal coactions

The framework of principal comodule algebras unifies in one category many algebraically constructed non-commutative examples and classical compact principal bundles.

Definition 4.1 ([BH04])**.** *Let $\mathcal{H}$ be a Hopf algebra with bijective antipode, and let $\Delta_{\mathcal{P}}\colon \mathcal{P}\to\mathcal{P}\otimes\mathcal{H}$ be a coaction making $\mathcal{P}$ an $\mathcal{H}$-comodule algebra. We call $\mathcal{P}$* principal *if and only if:*

1. *$\mathcal{P}\otimes_{\mathcal{B}}\mathcal{P}\ni p\otimes q\mapsto can(p\otimes q):=(p\otimes 1)\Delta_{\mathcal{P}}(q)\in\mathcal{P}\otimes\mathcal{H}$ is bijective, where $\mathcal{B}=\mathcal{P}^{\mathrm{co}\mathcal{H}}:=\{p\in\mathcal{P}\mid \Delta_{\mathcal{P}}(p)=p\otimes 1\}$;*
2. *there exists a left $\mathcal{B}$-linear right $\mathcal{H}$-colinear splitting of the multiplication map $\mathcal{B}\otimes\mathcal{P}\to\mathcal{P}$.*

Here (1) is the Hopf-Galois condition and (2) is right equivariant left projectivity of $\mathcal{P}$.

Alternatively, one can approach principality through strong connections:

Definition 4.2. *Let $\mathcal{H}$ be a Hopf algebra with bijective antipode S, and $\Delta_{\mathcal{P}}\colon\mathcal{P}\to\mathcal{P}\otimes\mathcal{H}$ be a coaction making $\mathcal{P}$ a right $\mathcal{H}$-comodule algebra. A* strong connection ℓ *on $\mathcal{P}$ is a unital linear map $\ell:\mathcal{H}\to\mathcal{P}\otimes\mathcal{P}$ satisfying:*

1. $(\mathrm{id}\otimes\Delta_{\mathcal{P}})\circ\ell=(\ell\otimes\mathrm{id})\circ\Delta$*;*
2. $({}_{\mathcal{P}}\Delta\otimes\mathrm{id})\circ\ell=(\mathrm{id}\otimes\ell)\circ\Delta$*, where* ${}_{\mathcal{P}}\Delta:=(S^{-1}\otimes\mathrm{id})\circ\mathrm{flip}\circ\Delta_{\mathcal{P}}$*;*
3. $\widetilde{can}\circ\ell=1\otimes\mathrm{id}$*, where* $\widetilde{can}\colon\mathcal{P}\otimes\mathcal{P}\ni p\otimes q\mapsto(p\otimes 1)\Delta_{\mathcal{P}}(q)\in\mathcal{P}\otimes\mathcal{H}$*.*

One can prove (see [BH] and references therein) that a comodule algebra is principal if and only if it admits a strong connection.

If $\Delta_M\colon M\to M\otimes C$ is a coaction making M a right comodule over a coalgebra C and N is a left C- comodule via a coaction ${}_N\Delta\colon N\to C\otimes N$, then we define their *cotensor product* as

$$M\underset{C}{\Box}N:=\{t\in M\otimes N\mid(\Delta_M\otimes\mathrm{id})(t)=(\mathrm{id}\otimes{}_N\Delta)(t)\}.\tag{4.12}$$

In particular, for a right $\mathcal{H}$-comodule algebra $\mathcal{P}$ and a left $\mathcal{H}$-comodule V, we observe that $\mathcal{P}\Box_{\mathcal{H}}V$ is a left $\mathcal{P}^{\mathrm{co}\,\mathcal{H}}$- module in a natural way. One of the key properties of principal comodule algebras is that, for any finite-dimensional left $\mathcal{H}$-comodule V, the left $\mathcal{P}^{\mathrm{co}\,\mathcal{H}}$-module $\mathcal{P}\Box_{\mathcal{H}}V$ is finitely generated projective [BH04]. Here $\mathcal{P}$ plays the role of a principal bundle and $\mathcal{P}\Box_{\mathcal{H}}V$ plays the role of an associated vector bundle. Therefore, we call $\mathcal{P}\Box_{\mathcal{H}}V$ an *associated module.*

Principality can also be characterized by the exactness and strong monoidality of the cotensor functor. This characterisation uses the notion of coflatness of a comodule: a right comodule is *coflat* if and only if cotensoring it with left comodules preserves exact sequences.

Theorem 4.3. *Let $\mathcal{H}$ be a Hopf algebra with bijective antipode, and $\mathcal{P}$ a right $\mathcal{H}$-comodule algebra. Then $\mathcal{P}$ is principal if and only if $\mathcal{P}$ is right $\mathcal{H}$-coflat and for all left $\mathcal{H}$-comodules V and W the map*

$$\beta:(\mathcal{P}\Box V)\underset{\mathcal{B}}{\otimes}(\mathcal{P}\Box W)\longrightarrow\mathcal{P}\Box(V\otimes W)$$
$$(a\otimes v)\otimes(b\otimes w)\longmapsto ab\otimes(v\otimes w)$$

is bijective. In other words, $\mathcal{P}$ is principal if and only if the cotensor product functor is exact and strongly monoidal with respect to the above map β.

Proof. The proof relies on putting together [S-HJ90, Theorem I], [S-P98, Theorem 6.15], [BH04, Theorem 2.5] and [SS05, Theorem 5.6]. First assume that $\mathcal{P}$ is principal. Then $\mathcal{P}$ is right equivariantly projective, and it follows from [BH04, Theorem 2.5] that $\mathcal{P}$ is faithfully flat. Now we can apply [S-P98, Theorem 6.15] to conclude that β is bijective. Furthermore, by [S-HJ90, Theorem I], the faithful flatness of $\mathcal{P}$ implies the coflatness of $\mathcal{P}$. Conversely, assume that cotensoring with $\mathcal{P}$ is exact and strongly monoidal with respect to β. Then substituting $\mathcal{H}$ for V and W yields the Hopf-Galois condition. Now [SS05, Theorem 5.6] implies the equivariant projectivity of $\mathcal{P}$. □

Appendix: Finite Galois coverings

Let X, Y be topological spaces and let $\pi\colon X \to Y$ be a covering map. Given any $y \in Y$, $\exists$ an open set U in Y with $y \in U$ such that $\pi^{-1}(U)$ is a disjoint union of open sets each of which π maps homeomorphically onto U. A *deck transformation* is a homeomorphism $h\colon X \to X$ with $\pi \circ h = \pi$.

Proposition A.4. *Let X and Y be compact Hausdorff topological spaces. Let $\pi\colon X \to Y$ be a covering map, and let Γ be the group of deck transformations of this covering. Assume that Γ is finite. Then X is a locally trivial principal Γ bundle on Y if and only if the canonical map*

$$can\colon C(X) \underset{C(Y)}{\otimes} C(X) \longrightarrow C(X) \otimes C(\Gamma)$$

$$can\colon f_1 \otimes f_2 \longmapsto (f_1 \otimes 1)\delta(f_2)$$

is an isomorphism.

Proof. Consider the commutative diagram

$$\begin{array}{ccc} C(X) \underset{C(Y)}{\otimes} C(X) & \xrightarrow{\;can\;} & C(X) \otimes C(\Gamma) \\ \downarrow & & \downarrow \\ C(X \underset{Y}{\times} X) & \longrightarrow & C(X \times \Gamma) \end{array} \tag{A.13}$$

in which each vertical arrow is the evident inclusion and the lower horizontal arrow is the $*$-homomorphism resulting from the map of topological spaces

$$X \times \Gamma \longrightarrow X \underset{Y}{\times} X, \qquad (x, \gamma) \mapsto (x, x\gamma). \tag{A.14}$$

Note that X is a (locally trivial) principal Γ bundle on Y if and only if this map of topological spaces is a homeomorphism, which is equivalent to the bijectivity of the lower horizontal arrow. Hence to prove the proposition, it will suffice to prove that the two vertical arrows are isomorphisms.

The right vertical arrow is an isomorphism because Γ is a finite group, so $C(\Gamma)$ is a finite dimensional vector space over the complex numbers $\mathbb{C}$. For the left vertical arrow, let E be the vector bundle on Y whose fiber at $y \in Y$ is $\mathrm{Map}(\pi^{-1}(y), \mathbb{C})$, i.e. is the set of all set-theoretic maps from $\pi^{-1}(y)$ to $\mathbb{C}$. Observe that $\pi^{-1}(y)$ is a discrete subset of the compact Hausdorff space X and therefore is finite.

Now, let $\mathcal{S}(E)$ be all the continuous sections of E. Then $\mathcal{S}(E) = C(X)$. Similarly, let

$$\rho\colon X \underset{Y}{\times} X \longrightarrow Y \quad \text{be} \quad (x_1, x_2) \mapsto \pi(x_1) = \pi(x_2), \tag{A.15}$$

and let F be the vector bundle on Y whose fiber at $y \in Y$ is $\mathrm{Map}(\rho^{-1}(y), \mathbb{C})$, i.e. is the set of all set-theoretic maps from $\rho^{-1}(y)$ to $\mathbb{C}$. Then $\mathcal{S}(F) = C(X \underset{Y}{\times} X)$, where $\mathcal{S}(F)$ is all the continuous sections of F. As vector bundles on Y, $F = E \otimes E$. This implies $\mathcal{S}(F) = \mathcal{S}(E) \underset{C(Y)}{\otimes} \mathcal{S}(E)$ and thus proves bijectivity for the left vertical arrow. □

Granted some connectivity conditions on X and Y (e.g. X and Y are connected finite CW complexes), it is then automatically the case that the group of deck transformations Γ is finite and that the action of Γ on X is free. The issue is then whether or not the action of Γ on each fiber of π is transitive. So a special case of the proposition is:

Proposition A.5. *Let X, Y be connected finite CW complexes. Let $\pi\colon X \to Y$ be a covering map. Γ denotes the group of deck transformations. Then the action of Γ on each fiber of π is transitive if and only if the canonical map*

$$can\colon C(X) \underset{C(Y)}{\otimes} C(X) \longrightarrow C(X) \otimes C(\Gamma) \tag{A.16}$$

is an isomorphism.

Without connectivity conditions the group of deck transformations can be infinite. Let Y be the Cantor set C and let X be $C \times \{0,1\}$ where the two-element set $\{0,1\}$ has the discrete topology. Let $\pi\colon C \times \{0,1\} \to C$ be the projection

$$\pi(c,t) = c \qquad c \in C \quad t \in \{0,1\}. \tag{A.17}$$

Let U be a subset of C which is both open and closed. Define $h_U\colon C \times \{0,1\} \to C \times \{0,1\}$ by

$$h_U(c,t) = \begin{cases} (c,t) & c \notin U \\ \\ (c, 1-t) & c \in U \end{cases} \tag{A.18}$$

Then h_U is a deck transformation and there are infinitely many h_U.

Acknowledgments. We thank Wojciech Szymański and Makoto Yamashita for helpful and enlightening discussions.

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