

On cocycle twisting of compact quantum groups

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Abstract

We provide a class of examples of compact quantum groups and unitary 2-cocycles on them, such that the twisted quantum groups are non-compact, but still locally compact quantum groups (in the sense of Kustermans and Vaes). This also gives examples of cocycle twists where the underlying C*-algebra of the quantum group changes.

Keywords: compact quantum group; von Neumann algebraic quantum group; cocycle twisting; comonoidal W*-Morita equivalence

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Introduction

In the seventies, Kac and Vainerman ([11]), and independently Enock and Schwartz ([6]), introduced the notion of (what was called by the latter) a *Kac algebra*, based on the fundamental work of Kac concerning ring groups in the sixties ([10]). Such Kac algebras, which are von Neumann algebras M with a coproduct $\Delta : M \rightarrow M \otimes M$ satisfying certain conditions, can naturally be made into a category, containing as a full sub-category the category of all locally compact groups, but allowing a duality functor which extends the Pontryagin duality functor on the sub-category of all *abelian* locally compact groups. Moreover, Kac algebras with a *commutative* underlying von Neumann algebra arise precisely from locally compact groups, by passing to the \mathcal{L}^∞ -space of the latter with respect to the left (or right) Haar measure, and with Δ dual to the multiplication in the group.

However, these Kac algebras do not cover all interesting examples of what could be called ‘locally compact quantum groups’. In the eighties, Woronowicz introduced ‘compact matrix quantum groups’ ([20]), which are to be seen as quantum versions of compact Lie groups. He also constructed in that paper certain compact matrix quantum groups $SU_q(2)$, which are deformations of the classical $SU(2)$ -group by some parameter $q \in \mathbb{R}$ with $0 < |q| < 1$. These quantum groups do not fit into the Kac algebra framework. The reason for this is that the antipode of these quantum groups is no longer a *-preserving anti-automorphism, but some unbounded operator on the associated C*-algebra of the quantum group.

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A satisfactory theory, covering both the compact quantum groups, the Kac algebras and some isolated examples, was obtained in 2000, when Kustermans and Vaes introduced their C*-algebraic quantum groups ([12]). In a follow-up paper, they also introduced von Neumann algebraic quantum groups ([13]), and proved that the C*-algebra approach and the von Neumann algebra approach were just different ways to look at the same structure (in that one can pass from the von Neumann algebra setup to a (reduced or universal) C*-algebraic setup and back). We also remark that in [18], an slightly alternative approach to von Neumann algebraic quantum groups was presented. In this paper, we will be mainly using the von Neumann algebraic approach, which simply asks for the existence of a coproduct and invariant weights on a von Neumann algebra (see Definition 1.2).

An interesting and important part of the theory consists in finding construction methods for von Neumann algebraic quantum groups. For example, in [1] the double product construction was worked out, while in [16] the bicrossed product construction was treated. In [4], we developed another construction method, namely the generalized twisting of a von Neumann algebraic quantum group (by a Galois object for its dual). This covers in particular the twisting by unitary 2-cocycles, special situations of which had been considered by Enock and Vainerman in [7], and by Fima and Vainerman in [8].

When we have two von Neumann algebraic quantum groups, one of which is obtained from the other by the above generalized twisting construction, we call them *comonoidally W^* -Morita equivalent*. The reason for this name is simple: the underlying von Neumann algebras of two such quantum groups are Morita equivalent (in the sense of Rieffel ([14])), with the equivalence ‘respecting the coproduct structure’. One can also show then that the representation categories of their associated universal C*-algebraic quantum groups are unitarily comonoidally equivalent (so that they have the ‘same’ tensor category of *-representations). The special case of cocycle twisting corresponds to those comonoidal Morita equivalences whose underlying Morita equivalence is (isomorphic to) the identity.

It was shown in [5] that comonoidal W^* -Morita equivalence provides a genuine equivalence relation between von Neumann algebraic quantum groups. It is then a natural question to find out which properties are preserved by this equivalence relation. In [2], it was shown for example that the discreteness of a quantum group is preserved, while its amenability is not. It also follows from the results of that paper that ‘being discrete and Kac’ is *not* preserved. In the general setting of von Neumann algebraic quantum groups, we showed in [5] that the scaling constant is an invariant for comonoidal W^* -Morita equivalence. In this article, we will show in a very concrete way that the notion of ‘being compact’ is *not* an invariant. This implies in particular that the representation category of a locally compact quantum group, as a monoidal W^* -category, does *not necessarily remember the topology of the quantum group*. In [5], we have also shown, by more general methods, that ‘being compact and Kac’ is an invariant.

This article is divided into two sections. *In the first section*, we recall some notions concerning compact quantum groups, von Neumann algebraic quantum groups and the twisting by unitary 2-cocycles. *In the second section*, we recall the definition of Woronowicz’s $SU_q(2)$ quantum groups. We then consider an infinite tensor product of these compact quantum groups over varying q ’s, with the condition that the q ’s go to zero sufficiently fast. Taking a limit of appropriate coboundaries, we obtain a 2-cocycle Ω on this infinite product quantum group which is no longer a coboundary. By giving an explicit formula for a non-finite but semi-finite invariant weight, we conclude that the Ω -twisted quantum group is no longer compact.

Remarks on notation: When A is a set, we denote by ι the identity map $A \rightarrow A$.

We will need the following tensor products: we denote by \odot the algebraic tensor product of two vector

spaces over \mathbb{C} , by $\underset{\min}{\otimes}$ the minimal tensor product of two C^* -algebras, and by \otimes the spatial tensor product between von Neumann algebras or Hilbert spaces.

We will further use the following notations concerning weights on von Neumann algebras. If M is a von Neumann algebra, and $\varphi : M^+ \rightarrow [0, +\infty]$ an nsf (= normal semi-finite faithful) weight, we denote by $\mathcal{N}_\varphi \subseteq M$ the σ -weakly dense left ideal of square integrable elements:

$$\mathcal{N}_\varphi = \{x \in M \mid \varphi(x^*x) < \infty\}.$$

We denote by $\mathcal{L}^2(M, \varphi)$ the Hilbert space completion of \mathcal{N}_φ with respect to the inner product

$$\langle x, y \rangle = \varphi(y^*x),$$

and by Λ_φ the canonical embedding $\mathcal{N}_\varphi \rightarrow \mathcal{L}^2(M, \varphi)$. We remark that Λ_φ is $(\sigma$ -strong)-(norm) closed. We denote \mathcal{M}_φ^+ for the space of elements $x \in M^+$ for which $\varphi(x) < \infty$, and \mathcal{M}_φ for the complex linear span of \mathcal{M}_φ^+ . One can show then that $\mathcal{M}_\varphi = \mathcal{N}_\varphi^* \cdot \mathcal{N}_\varphi$, i.e. any element x of \mathcal{M}_φ can be written as $\sum_{i=1}^n x_i^* y_i$ with $x_i, y_i \in \mathcal{N}_\varphi$.

As far as applicable, we use the same notation when considering, more generally, operator valued weights. We also remark here that if M_1 and M_2 are von Neumann algebras, and φ an nsf weight on M_2 , we can make sense of $(\iota \otimes \varphi)$ as an nsf operator valued weight from $M_1 \otimes M_2$ to M_1 in a natural way: if $x \in (M_1 \otimes M_2)^+$, we let $(\iota \otimes \varphi)(x)$ be the element

$$\omega \in (M_1)_*^+ \rightarrow \varphi((\omega \otimes \iota)(x)) \in [0, +\infty]$$

in the extended positive cone of M_1 . For more information concerning the theory of weights and operator valued weights, we refer to the first chapters of [15].

1 Compact and von Neumann algebraic quantum groups

As we mentioned in the introduction, S.L. Woronowicz developed the notion of a compact matrix quantum group in [20] (there called compact matrix *pseudogroup*). Later, he also introduced the more general notion of a compact quantum group ([22]).

Definition 1.1. A compact quantum group consists of a couple (A, Δ) , where A is a unital C^* -algebra, and Δ a unital $*$ -homomorphism $A \rightarrow A \underset{\min}{\otimes} A$ such that

1. the map Δ is coassociative:

$$(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta,$$

and

2. the linear subspaces

$$\Delta(A)(1 \otimes A) := \left\{ \sum_i \Delta(a_i)(1 \otimes b_i) \mid a_i, b_i \in A \right\}$$

and

$$\Delta(A)(A \otimes 1) := \left\{ \sum_i \Delta(a_i)(b_i \otimes 1) \mid a_i, b_i \in A \right\}$$

are norm-dense in $A \underset{\min}{\otimes} A$.

The compact quantum group is called a compact matrix quantum group if there exists $n \in \mathbb{N}_0$ and a unitary $u = \sum_{i,j=1}^n u_{ij} \otimes e_{ij} \in A \otimes M_n(\mathbb{C})$, such that $\Delta(u_{ij}) = \sum_{k=1}^n u_{ik} \otimes u_{kj}$ and such that the u_{ij} generate A as a unital C^* -algebra. Such a unitary is then called a fundamental unitary corepresentation.

It is not difficult to show that any compact quantum group (A, Δ) with A commutative is of the form $(C(\mathfrak{G}), \Delta)$ for some compact group \mathfrak{G} , and Δ dual to the group multiplication. Moreover, (A, Δ) will then be a compact matrix quantum group iff \mathfrak{G} is a compact Lie group. It is common practice to denote, by analogy, also a general compact quantum group (A, Δ) as $(C(\mathfrak{G}), \Delta)$, although this notation is now of course purely formal, since there is no underlying object \mathfrak{G} .

In [22] (and [17] for the non-separable case), it is proven that to any compact quantum group $(C(\mathfrak{G}), \Delta)$ one can associate a unique state φ satisfying

$$(\iota \otimes \varphi)\Delta(a) = \varphi(a)1 = (\varphi \otimes \iota)\Delta(a), \quad \forall a \in C(\mathfrak{G}).$$

This state is called the *invariant state* of the compact quantum group. It is also proven there that with any compact quantum group, one can associate a Hopf $*$ -algebra $(\text{Pol}(\mathfrak{G}), \Delta)$ such that $\text{Pol}(\mathfrak{G}) \subseteq C(\mathfrak{G})$ is a norm-dense sub- $*$ -algebra, the comultiplication being the restriction of the comultiplication on $C(\mathfrak{G})$. The invariant state is then faithful on $\text{Pol}(\mathfrak{G})$. Conversely, any Hopf $*$ -algebra $(\text{Pol}(\mathfrak{G}), \Delta)$ possessing an invariant state can be completed to a compact quantum group in essentially two ways. The first construction gives the associated *reduced* compact quantum group. Its underlying C^* -algebra $C_r(\mathfrak{G})$ is given as the closure of the image of the GNS-representation of $\text{Pol}(\mathfrak{G})$ with respect to the invariant state. The second construction gives the associated *universal* compact quantum group. Its underlying C^* -algebra $C_u(\mathfrak{G})$ is now the universal C^* -envelope of $\text{Pol}(\mathfrak{G})$ (which can be shown to exist). For coamenable compact quantum groups, which are those compact quantum groups possessing both a bounded counit and a *faithful* invariant state, the reduced and universal construction for the underlying Hopf $*$ -algebra both coincide with the original compact quantum group, so that in this case, one only has to specify the Hopf $*$ -algebra to determine completely the associated C^* -algebraic structure.

With any compact quantum group $C(\mathfrak{G})$, one can also associate a von Neumann algebra, which we will denote as $\mathcal{L}^\infty(\mathfrak{G})$. It is the σ -weak closure of the image of $\text{Pol}(\mathfrak{G})$ under the GNS-representation for φ . Then Δ can be completed to a normal unital $*$ -homomorphism $\Delta : \mathcal{L}^\infty(\mathfrak{G}) \rightarrow \mathcal{L}^\infty(\mathfrak{G}) \otimes \mathcal{L}^\infty(\mathfrak{G})$. This makes $(\mathcal{L}^\infty(\mathfrak{G}), \Delta)$ into a von Neumann algebraic quantum group, whose definition we now present.

Definition 1.2. ([13]) *A von Neumann algebraic quantum group is a couple (M, Δ) , consisting of a von Neumann algebra M and a normal unital $*$ -homomorphism $\Delta : M \rightarrow M \otimes M$, such that*

1. *the map Δ is coassociative:*

$$(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta,$$

and

2. *there exist normal, semi-finite, faithful (nsf) weights φ and ψ on M such that, for any state $\omega \in M_*$, we have*

$$\varphi((\omega \otimes \iota)\Delta(x)) = \varphi(x), \quad \forall x \in \mathcal{M}_\varphi^+$$

and

$$\psi((\iota \otimes \omega)\Delta(x)) = \psi(x), \quad \forall x \in \mathcal{M}_\psi^+.$$

The weights appearing in this definition turn out to be unique (up to multiplication with a positive non-zero scalar), and are called respectively the *left and right invariant weights*.

The von Neumann algebraic quantum groups $(\mathcal{L}^\infty(\mathfrak{G}), \Delta)$ arising from compact quantum groups can be characterized as those von Neumann algebraic quantum groups (M, Δ) which have a left invariant normal *state*. One can also show that from such a $(\mathcal{L}^\infty(\mathfrak{G}), \Delta)$, a σ -weakly dense Hopf $*$ -subalgebra $(\text{Pol}(\mathfrak{G}), \Delta)$ can be reconstructed, providing a one-to-one correspondence between von Neumann algebraic quantum groups with an invariant normal state and Hopf $*$ -algebras with an invariant state.

We now introduce the notion of a unitary 2-cocycle.

Definition 1.3. *Let (M, Δ) be a von Neumann algebraic quantum group. A unitary 2-cocycle for (M, Δ) is a unitary element $\Omega \in M \otimes M$ satisfying the 2-cocycle equation*

$$(\Omega \otimes 1)(\Delta \otimes \iota)(\Omega) = (1 \otimes \Omega)(\iota \otimes \Delta)(\Omega).$$

It is then easily seen that if (M, Δ) is a von Neumann algebraic quantum group, and Ω a unitary 2-cocycle for it, we can define a new coproduct Δ_Ω on M by putting

$$\Delta_\Omega(x) := \Omega \Delta(x) \Omega^*, \quad \forall x \in M. \quad (1)$$

The coassociativity of Δ_Ω then follows precisely from the 2-cocycle equation. A non-trivial result from [4] states that (M, Δ_Ω) in fact possesses invariant nsf weights, so that it is a von Neumann algebraic quantum group. The construction of these weights is rather complicated, and relies on some deep theorems from non-commutative integration theory. However, in the concrete example which we develop in the next section, we will be able to construct the weights on our cocycle twisted quantum group in a fairly straightforward way.

2 Twisting does not preserve compactness

The compact quantum groups which we will need will be constructed using the ‘twisted $SU(2)$ ’ groups from [20] (see also [21]). We recall their definition.

Definition-Proposition 2.1. *Let q be a real number with $0 < |q| \leq 1$. Define $\text{Pol}(SU_q(2))$ as the unital $*$ -algebra, generated (as a unital $*$ -algebra) by two generators a and b satisfying the relations*

$$\begin{cases} a^*a + b^*b = 1 & ab = qba \\ aa^* + q^2bb^* = 1 & a^*b = q^{-1}ba^* \\ bb^* = b^*b. \end{cases}$$

Then there exists a Hopf $$ -algebra structure $(\text{Pol}(SU_q(2)), \Delta)$ on $\text{Pol}(SU_q(2))$ which satisfies*

$$\begin{cases} \Delta(a) = a \otimes a - qb^* \otimes b \\ \Delta(b) = b \otimes a + a^* \otimes b. \end{cases}$$

Moreover, this Hopf $$ -algebra possesses an invariant state φ , and has a unique completion to a compact matrix quantum group $(C(SU_q(2)), \Delta)$.*

We will always use a and b to denote the generators of $C(SU_q(2))$. Later on however, when we will be working with a sequence of $SU_{q_n}(2)$ ’s, we will index these generators by the corresponding index n of q_n . We will follow the same convention for the comultiplication, the invariant state, and the other special elements in $C(SU_q(2))$ which we will later introduce.

The following proposition gives us more information about how the underlying C^* -algebra and the associated invariant state of $C(SU_q(2))$ look like.

Proposition 2.2. ([21]) Let q be a real number with $0 < |q| < 1$. Let \mathcal{H} be the Hilbert space $l^2(\mathbb{N}) \otimes l^2(\mathbb{Z})$, whose canonical basis elements we denote as $\xi_{n,k}$ (and with the convention $\xi_{n,k} = 0$ when $n < 0$). Then there exists a faithful unital $*$ -representation of $C(SU_q(2))$ on \mathcal{H} , determined by

$$\begin{cases} \pi(a)\xi_{n,k} = \sqrt{1 - q^{2n}}\xi_{n-1,k}, \\ \pi(b)\xi_{n,k} = q^n\xi_{n,k+1}. \end{cases}$$

The invariant state φ on $C(SU_q(2))$ is given by

$$\varphi(x) = (1 - q^2) \sum_{n \in \mathbb{N}} q^{2n} \langle \pi(x)\xi_{n,0}, \xi_{n,0} \rangle.$$

We now introduce special elements which will be of importance later on. Define, in the notation of the previous proposition, matrix units f_{kl} on $l^2(\mathbb{N}) \otimes l^2(\mathbb{Z})$, by putting $f_{mn}\xi_{r,k} = \delta_{r,n}\xi_{m,k}$, with δ the Kronecker delta, and where m, n take values in \mathbb{N} . It is clear then that each f_{mn} is in the unital C^* -algebra generated by $\pi(a)$. Hence $e_{mn} := \pi^{-1}(f_{mn})$ is an element of $C(SU_q(2))$. We define elements $p, p', w \in C(SU_q(2))$ by the following formulas:

$$p := e_{00}, \tag{2}$$

$$p' := e_{11}, \tag{3}$$

$$w := e_{01} + e_{12} + e_{20} + \sum_{k=3}^{\infty} e_{kk}. \tag{4}$$

Thus, with respect to the matrix units e_{mn} , the element w is the unitary

$$w = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix}.$$

We need to form infinite tensor products of the above quantum groups (see [19] for more detailed information). Let $\mathbf{q} := (q_n)_{n \in \mathbb{N}_0}$ be a sequence of reals satisfying $0 < |q_n| < 1$. Then we can form an inductive sequence

$$\bigodot_{k=1}^n (\text{Pol}(SU_{q_k}(2)), \Delta_k)$$

of Hopf $*$ -algebras, by tensoring with 1 to the right at each step. We denote the inductive limit as $(\text{Pol}(SU_{\mathbf{q}}(2)), \Delta_{\mathbf{q}})$. It is easy to see that this is again a Hopf $*$ -algebra with an invariant state $\varphi_{\mathbf{q}}$, which is the pointwise limit of the functionals $\bigodot_{k=1}^n \varphi_k$. The associated compact quantum group will then again be coamenable, with underlying C^* -algebra $C(SU_{\mathbf{q}}(2))$ the universal infinite tensor product of the $C(SU_{q_k}(2))$. This will equal the reduced C^* -tensor product with respect to the states φ_k , by nuclearity of the $C(SU_{q_k}(2))$ (see the appendix of [21]).

We will show now that those $(C(SU_{\mathbf{q}}(2)), \Delta_{\mathbf{q}})$ for which \mathbf{q} is square summable possess the property enunciated in the abstract, namely: they possess a unitary 2-cocycle which allows to twist them into a non-compact quantum group. Therefore, *we now fix some \mathbf{q} satisfying the property of square summability.*

We need to show some properties of $\mathcal{L}^\infty(SU_{\mathbf{q}}(2))$. We begin with some well-known general lemmas.

Lemma 2.3. *Let $M \subseteq B(\mathcal{H})$ be a von Neumann algebra, and $\xi \in \mathcal{H}$ a separating vector for M . Let x_n be a bounded sequence in M for which $x_n \xi$ converges to a vector η . Then x_n converges in the σ -strong topology.*

Lemma 2.4. *([9], Proposition 1.1) Let (\mathcal{H}_n, ξ_n) be a sequence of Hilbert spaces with distinguished unit vectors, and let (\mathcal{H}, ξ) be their infinite tensor product. Let $\eta_n \in \mathcal{H}_n$ be non-zero vectors with $\|\eta_n\| \leq 1$. Then $(\bigotimes_{k=1}^n \eta_k) \otimes (\bigotimes_{k=n+1}^{\infty} \xi_k) \in \mathcal{H}$ converges to a non-zero vector $\bigotimes_{k=1}^{\infty} \eta_k$ if $\sum |1 - \langle \eta_k, \xi_k \rangle| < \infty$.*

Recall the special element p (defined at (2)), for which we will now also use index notation.

Lemma 2.5. *The sequence $(\bigotimes_{k=1}^n p_k) \otimes 1$ converges σ -strongly in $\mathcal{L}^{\infty}(SU_{\mathbf{q}}(2))$ to an operator $\bigotimes_{k=1}^{\infty} p_k$, while the sequence $1 \otimes (\bigotimes_{k=n+1}^{\infty} p_k)$ converges σ -strongly to 1.*

Proof. Note first that the GNS construction of $\mathcal{L}^{\infty}(SU_{\mathbf{q}}(2))$ with respect to $\varphi_{\mathbf{q}}$ can be identified with $\bigotimes_{k=1}^{\infty} (\mathcal{L}^2(SU_{q_k}(2)), \xi_k)$, with ξ_k the separating and cyclic vector associated to φ_k . Combining Lemma 2.3 and Lemma 2.4, we only have to see if $\sum (1 - \varphi_k(p_k)) < \infty$ to have the first statement of the lemma. Since $\varphi_k(p_k) = 1 - q_k^2$, this follows by the assumption that q_k is a square summable sequence.

Then we can of course also make sense of $x \otimes (\bigotimes_{k=n+1}^{\infty} p_k)$, for any $x \in \bigotimes_{k=1}^n \mathcal{L}^{\infty}(SU_{q_k}(2))$. Since

$$\begin{aligned} \left\| \left(\bigotimes_{k=1}^n \xi_{q_k} \right) \otimes \left(\bigotimes_{k=n+1}^{\infty} p_k \xi_{q_k} \right) - \bigotimes_{k=1}^{\infty} \xi_{q_k} \right\|^2 &= 1 - \prod_{k=n+1}^{\infty} \varphi_{q_k}(p_k) \\ &= 1 - \prod_{k=n+1}^{\infty} (1 - q_k^2), \end{aligned}$$

the second statement follows from the convergence of $\prod_{k=1}^{\infty} (1 - q_k^2)$ to a non-zero number, which in turn follows easily from the square summability of the q_n . \square

Corollary 2.6. *Denote $E_n(x) = s_n x s_n$, where $s_n = 1 \otimes (\bigotimes_{k=n+1}^{\infty} p_k)$ and $x \in \mathcal{L}^{\infty}(SU_{\mathbf{q}}(2))$. Then $E_n(x)$ converges to x in the σ -strong topology.*

Recall the special elements w defined by the formula (4). We again denote this element, when we regard it inside some $C(SU_{q_n}(2))$, as w_n .

Lemma 2.7. *Let $0 < |q| < 1$. Let ξ be the cyclic separating vector in the GNS-construction for $\mathcal{L}^{\infty}(SU_q(2))$ w.r.t. the invariant state φ . Then*

$$\|(w^* - a^*)\xi\| \leq 3q^2.$$

Proof. In the matrix representation introduced just after Proposition 2.2, it is easy to calculate that

$$(w - a)(w^* - a^*) = \begin{pmatrix} (1 - \sqrt{1 - q^2})^2 & 0 & 0 & 0 & 0 & \dots \\ 0 & (1 - \sqrt{1 - q^4})^2 & 0 & 0 & 0 & \dots \\ 0 & 0 & 2 - q^6 & -\sqrt{1 - q^6} & 0 & \dots \\ 0 & 0 & -\sqrt{1 - q^6} & 2 - q^8 & -\sqrt{1 - q^8} & \dots \\ 0 & 0 & 0 & -\sqrt{1 - q^8} & 2 - q^{10} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Since $1 - \sqrt{1-c} \leq c$ for $0 \leq c \leq 1$, we have

$$\begin{aligned}
& \varphi((w-a)(w^*-a^*)) \\
&= (1-q^2)((1-\sqrt{1-q^2})^2 + q^2(1-\sqrt{1-q^4})^2 + q^4(2-q^6) + q^6(2-q^8) + \dots) \\
&\leq 2(1-q^2)(q^4 + q^{10} + q^4 + q^6 + \dots) \\
&\leq 9q^4.
\end{aligned}$$

So $\|(w^*-a^*)\xi\| \leq 3q^2$. □

Theorem 2.8. *The σ -strong* limit $\bigotimes_{n=1}^{\infty} ((w_n \otimes w_n)\Delta_n(w_n^*))$ exists in $\mathcal{L}^{\infty}(SU_{\mathbf{q}}(2)) \otimes \mathcal{L}^{\infty}(SU_{\mathbf{q}}(2))$, and determines a unitary 2-cocycle Ω for $\mathcal{L}^{\infty}(SU_{\mathbf{q}}(2))$.*

Proof. Remark that $(\mathcal{L}^{\infty}(SU_{\mathbf{q}}(2)) \otimes \mathcal{L}^{\infty}(SU_{\mathbf{q}}(2)), \varphi_{\mathbf{q}} \otimes \varphi_{\mathbf{q}})$ can be identified with

$$\left(\bigotimes_{k=1}^{\infty} (\mathcal{L}^{\infty}(SU_{q_k}(2)) \otimes (\mathcal{L}^{\infty}(SU_{q_k}(2)))) \right), \bigotimes_{k=1}^{\infty} (\varphi_k \otimes \varphi_k).$$

Then by the lemmas 2.3 and 2.4, it is enough to prove that

$$\sum_{n=1}^{\infty} |1 - (\varphi_n \otimes \varphi_n)((w_n \otimes w_n)\Delta_n(w_n^*))| < \infty$$

to know that $\bigotimes_{k=1}^n ((w_k \otimes w_k)\Delta_k(w_k^*))$ converges in the σ -strong*-topology.

Denoting by ξ_k the GNS vector associated with φ_k , we estimate

$$\begin{aligned}
& \sum_{n=1}^{\infty} |1 - (\varphi_n \otimes \varphi_n)((w_n \otimes w_n)\Delta_n(w_n^*))| \\
&\leq \sum_{n=1}^{\infty} |1 - (\varphi_n \otimes \varphi_n)((a_n \otimes a_n)\Delta_n(a_n^*))| \\
&\quad + \sum_{n=1}^{\infty} |(\varphi_n \otimes \varphi_n)((a_n \otimes a_n)(\Delta_n(a_n^*) - \Delta_n(w_n^*)))| \\
&\quad + \sum_{n=1}^{\infty} |(\varphi_n \otimes \varphi_n)((a_n \otimes a_n - w_n \otimes w_n)\Delta_n(w_n^*))| \\
&\leq \sum_{n=1}^{\infty} |1 - (\varphi_n \otimes \varphi_n)((a_n \otimes a_n)\Delta_n(a_n^*))| \\
&\quad + \sum_{n=1}^{\infty} |(\varphi_n \otimes \varphi_n)(\Delta_n((a_n - w_n)(a_n - w_n)^*))|^{1/2} \\
&\quad + \sum_{n=1}^{\infty} |(\varphi_n \otimes \varphi_n)((a_n \otimes a_n - w_n \otimes w_n)(a_n \otimes a_n - w_n \otimes w_n)^*)|^{1/2} \\
&= \sum_{n=1}^{\infty} |1 - (\varphi_n \otimes \varphi_n)((a_n \otimes a_n)\Delta_n(a_n^*))| \\
&\quad + \sum_{n=1}^{\infty} \|(a_n - w_n)^*\xi_n\| + \sum_{n=1}^{\infty} \|(a_n \otimes a_n - w_n \otimes w_n)^*(\xi_n \otimes \xi_n)\| \\
&\leq \sum_{n=1}^{\infty} |1 - (\varphi_n \otimes \varphi_n)((a_n \otimes a_n)\Delta_n(a_n^*))| + 3 \sum_{n=1}^{\infty} \|(a_n - w_n)^*\xi_n\|.
\end{aligned}$$

By the previous lemma and the square summability of the q_n , we have

$$\sum_{n=1}^{\infty} \|(a_n - w_n)^* \xi_n\| < \infty.$$

So we only have to compute if

$$\sum_{n=1}^{\infty} |1 - (\varphi_n \otimes \varphi_n)((a_n \otimes a_n)\Delta_n(a_n^*))| < \infty.$$

Now an easy calculation shows that

$$(\varphi_n \otimes \varphi_n)((a_n \otimes a_n)\Delta_n(a_n^*)) = \frac{1}{(1 + q_n^2)^2}.$$

Since $1 - \frac{1}{(1+q_n^2)^2} \leq 2q_n^2$, we can again conclude convergence by square summability of the q_n .

Thus

$$\Omega = \bigotimes_{n=1}^{\infty} ((w_n \otimes w_n)\Delta_n(w_n^*))$$

is a well-defined unitary as a σ -strong* limit of unitaries. Since multiplication is jointly continuous on the group of unitaries with the σ -strong* topology, Ω will satisfy the 2-cocycle identity since each $\bigotimes_{k=1}^n ((w_k \otimes w_k)\Delta_k(w_k^*))$ does. □

Lemma 2.9. *Let $0 < |q| < 1$, and take $w \in C(SU_q(2))$ as defined by the formula (4). Then the invariant state φ for $C(SU_q(2))$ satisfies*

$$\varphi \leq q^{-2} \varphi(w^* \cdot w).$$

Proof. This follows from a straightforward computation, using the concrete form of φ as in Proposition 2.2. □

Hence we can form on $\mathcal{L}^\infty(SU_q(2))$ the normal faithful weight

$$\varphi_\Omega := \lim_{n \rightarrow \infty} \left(\prod_{k=1}^n q_k^{-2} \left(\bigotimes_{k=1}^n \varphi_k(w_k^* \cdot w_k) \right) \otimes \left(\bigotimes_{k=n+1}^{\infty} \varphi_k \right) \right),$$

the limit being taken pointwise on elements of $\mathcal{L}^\infty(SU_q(2))^+$. This is a well-defined normal faithful weight, since it is an increasing sequence of normal, faithful, positive functionals. It is clear that φ_Ω is not finite.

Proposition 2.10. *The weight φ_Ω on $\mathcal{L}^\infty(SU_q(2))$ is semi-finite.*

Proof. We prove that the projections $1 \otimes \left(\bigotimes_{k=n+1}^{\infty} p_k \right)$, introduced in Lemma 2.5, are integrable with respect to φ_Ω . By Corollary 2.6, this will prove the proposition.

But $w_n^* p_n w_n = p'_n$, using also the notation of (3). Since $\varphi_n(p'_n) = q_n^2(1 - q_n^2)$, the integrability of all $1 \otimes \left(\bigotimes_{k=n+1}^{\infty} p_k \right)$ follows. □

We now end by proving that φ_Ω is an invariant nsf weight for the couple $(\mathcal{L}^\infty(SU_{\mathbf{q}}(2)), \Delta_\Omega)$, where Ω was introduced in Theorem 2.8 and φ_Ω just before Lemma 2.10, and where Δ_Ω is the twisted coproduct defined by (1).

Theorem 2.11. *The nsf weight φ_Ω is a left and right invariant nsf weight for $(\mathcal{L}^\infty(SU_{\mathbf{q}}(2)), \Delta_\Omega)$.*

Proof. We only prove left invariance, since the proof for right invariance follows by symmetry.

Denote

$$\mathcal{P}_n = \{x \otimes (\bigotimes_{k=n+1}^{\infty} p_k) \mid x \in \bigotimes_{k=1}^n \mathcal{L}^\infty(SU_{q_k}(2))\},$$

still using the notation (2), and denote $\mathcal{P} = \bigcup_{n \in \mathbb{N}_0} \mathcal{P}_n$. By the proof of Lemma 2.10, we know that \mathcal{P} consists of integrable elements for φ_Ω . We first show that also $\Delta_\Omega(\mathcal{P}) \subseteq \mathcal{M}_{(\iota \otimes \varphi_\Omega)}$, and that

$$(\iota \otimes \varphi_\Omega)(\Delta_\Omega(y)) = \varphi_\Omega(y)1$$

for y in \mathcal{P} .

Choose $x = \bigotimes_{k=1}^n x_n \in \bigotimes_{k=1}^n \mathcal{L}^\infty(SU_{q_k}(2))$ with all x_n positive, and put $y = x \otimes (\bigotimes_{k=n+1}^{\infty} p_k)$. Then $(\iota \otimes \varphi_\Omega)(\Delta_\Omega(y))$ is an element in the extended positive cone of $\mathcal{L}^\infty(SU_{\mathbf{q}}(2))$. As such, it is the pointwise supremum of the elements

$$z_m = (\bigotimes_{k=1}^n \alpha_k) \otimes (\bigotimes_{k=n+1}^{n+m} \beta_k) \otimes (\bigotimes_{k=n+m+1}^{\infty} \gamma_k),$$

regarded as semi-linear functionals on $\mathcal{L}^\infty(SU_{\mathbf{q}}(2))_*$, where

$$\alpha_k = q_k^{-2}(\iota \otimes \varphi_k(w_k^* \cdot w_k))((w_k \otimes w_k)\Delta_k(w_k^* x_k w_k)(w_k^* \otimes w_k^*)),$$

$$\beta_k = q_k^{-2}(\iota \otimes \varphi_k(w_k^* \cdot w_k))((w_k \otimes w_k)\Delta_k(w_k^* p_k w_k)(w_k^* \otimes w_k^*))$$

and

$$\gamma_k = (\iota \otimes \varphi_k)((w_k \otimes w_k)\Delta_k(w_k^* p_k w_k)(w_k^* \otimes w_k^*)),$$

and where the infinite tensor product is to be seen as a σ -strong limit.

We can simplify α_k and β_k to respectively $q_k^{-2}\varphi_k(w_k^* x_k w_k)1$ and $q_k^{-2}\varphi_k(w_k^* p_k w_k)1$, by invariance of φ_k , while γ_k satisfies

$$\gamma_k \leq q_k^{-2}\varphi_k(w_k^* p_k w_k)1$$

by Lemma 2.9 and left invariance of φ_k . This implies that $z_m \leq \varphi_\Omega(y)1$. Since z_m is increasing, and the invariant state $\varphi_{\mathbf{q}}$ on $\mathcal{L}^\infty(SU_{\mathbf{q}}(2))$ is a faithful normal state, we only have to prove that $\varphi_{\mathbf{q}}(z_m) \rightarrow \varphi_\Omega(y)$ to conclude that $(\iota \otimes \varphi_\Omega)(\Delta_\Omega(y)) = \varphi_\Omega(y)1$.

But it is easily seen that $\varphi_{\mathbf{q}}(z_m)$ converges to $\varphi_\Omega(y)$ if we can show that

$$\prod_{k=n+m+1}^{\infty} \varphi_k(\gamma_k) \xrightarrow{m \rightarrow \infty} 1.$$

This is equivalent with proving that

$$\prod_1^{\infty} (\varphi_k \otimes \varphi_k)((w_k \otimes w_k)\Delta_k(w_k^* p_k w_k)(w_k^* \otimes w_k^*)) \neq 0.$$

Since the left hand side equals $(\varphi_{\mathbf{q}} \otimes \varphi_{\mathbf{q}})(\Omega \Delta_{\mathbf{q}}(s)\Omega^*)$, with $s = \bigotimes_{k=1}^{\infty} p_k$, the above product is indeed non-zero, by faithfulness of $\varphi_{\mathbf{q}}$.

One then easily concludes that, since any $y \in \mathcal{P}$ is a linear combination of elements of the above form, we have $\Delta_{\Omega}(y) \in \mathcal{M}_{\iota \otimes \varphi_{\Omega}}$ for $y \in \mathcal{P}$, with $(\iota \otimes \varphi_{\Omega})\Delta_{\Omega}(y) = \varphi_{\Omega}(y)$.

Next we prove that \mathcal{P} is a σ -strong-norm core for the GNS-map $\Lambda_{\varphi_{\Omega}}$ associated with the nsf weight φ_{Ω} . For this, it is enough to prove that $s_n = 1 \otimes (\bigotimes_{k=n+1}^{\infty} p_k)$ is invariant under $\sigma_t^{\varphi_{\Omega}}$ for any $t \in \mathbb{R}$, since then, using Corollary 2.6, we can conclude that, whenever $y \in \mathcal{N}_{\varphi_{\Omega}}$, we will have $s_n y s_n \rightarrow y$ in the σ -strong topology and $\Lambda_{\varphi_{\Omega}}(s_n y s_n) = s_n J_{\varphi_{\Omega}} s_n J_{\varphi_{\Omega}} \Lambda_{\varphi_{\Omega}}(y) \rightarrow \Lambda_{\varphi_{\Omega}}(y)$ in the norm topology (where $J_{\varphi_{\Omega}}$ is the modular conjugation on $\mathcal{L}^2(M, \varphi_{\Omega})$).

By formula (A 1.4) of [20] (or by a direct verification), we have that $\sigma_t^{\varphi_k}(a_k) = |q|^{2it} a_k$ for any $t \in \mathbb{R}$. Hence $a_k a_k^*$ is in the centralizer of φ_k , and so the same is true of p_k and p_k' . Since $s_n \in \mathcal{M}_{\varphi_{\Omega}}$, we will have $x s_n, s_n x \in \mathcal{M}_{\varphi_{\Omega}}$ for $x \in \mathcal{N}_{\varphi_{\Omega}} \cap \mathcal{N}_{\varphi_{\Omega}}^*$, and then

$$\begin{aligned} \varphi_{\Omega}(s_n x) &= \lim_{m \rightarrow \infty} \left(\prod_{k=1}^{n+m} q_k^{-2} \right) \left(\left(\bigotimes_{k=1}^n \varphi_k(w_k^* \cdot w_k) \right) \otimes \left(\bigotimes_{k=n+1}^{n+m} \varphi_k(w_k^* p_k \cdot w_k) \right) \otimes \left(\bigotimes_{k=n+m+1}^{\infty} \varphi_k(p_k \cdot) \right) \right)(x) \\ &= \lim_{m \rightarrow \infty} \left(\prod_{k=1}^{n+m} q_k^{-2} \right) \left(\left(\bigotimes_{k=1}^n \varphi_k(w_k^* \cdot w_k) \right) \otimes \left(\bigotimes_{k=n+1}^{n+m} \varphi_k(p_k' w_k^* \cdot w_k) \right) \otimes \left(\bigotimes_{k=n+m+1}^{\infty} \varphi_k(p_k \cdot) \right) \right)(x) \\ &= \lim_{m \rightarrow \infty} \left(\prod_{k=1}^{n+m} q_k^{-2} \right) \left(\left(\bigotimes_{k=1}^n \varphi_k(w_k^* \cdot w_k) \right) \otimes \left(\bigotimes_{k=n+1}^{n+m} \varphi_k(w_k^* \cdot w_k p_k') \right) \otimes \left(\bigotimes_{k=n+m+1}^{\infty} \varphi_k(\cdot p_k) \right) \right)(x) \\ &= \varphi_{\Omega}(x s_n). \end{aligned}$$

From this, the equality $\sigma_t^{\varphi_{\Omega}}(s_n) = s_n$ for any $t \in \mathbb{R}$ follows (for example by Theorem VIII.2.6 of [15]).

We can now conclude the proof. By left-invariance of φ_{Ω} on \mathcal{P} , we can introduce an isometry W_{Ω}^* on $\mathcal{L}^2(SU_{\mathbf{q}}(2), \varphi_{\Omega}) \otimes \mathcal{L}^2(SU_{\mathbf{q}}(2), \varphi_{\Omega})$ by putting

$$W_{\Omega}^*(\Lambda_{\varphi_{\Omega}}(x) \otimes \Lambda_{\varphi_{\Omega}}(y)) = (\Lambda_{\varphi_{\Omega}} \otimes \Lambda_{\varphi_{\Omega}})(\Delta_{\Omega}(y)(x \otimes 1))$$

for $x \in \mathcal{N}_{\varphi_{\Omega}}$ and $y \in \mathcal{P}$. Then by the core-property of \mathcal{P} , we conclude that for $x \in \mathcal{N}_{\varphi_{\Omega}}$ and any $y \in \mathcal{N}_{\varphi_{\Omega}}$, we have $\Delta_{\Omega}(y)(x \otimes 1)$ square integrable for $\varphi_{\Omega} \otimes \varphi_{\Omega}$, with

$$W_{\Omega}^*(\Lambda_{\varphi_{\Omega}}(x) \otimes \Lambda_{\varphi_{\Omega}}(y)) = (\Lambda_{\varphi_{\Omega}} \otimes \Lambda_{\varphi_{\Omega}})(\Delta_{\Omega}(y)(x \otimes 1)).$$

Hence

$$\varphi_{\Omega}((\omega \otimes \iota)\Delta_{\Omega}(y)) = \varphi_{\Omega}(y)$$

for $y \in \mathcal{M}_{\varphi_{\Omega}}^+$ and ω a state of the form $\langle \cdot \Lambda_{\varphi_{\Omega}}(x), \Lambda_{\varphi_{\Omega}}(x) \rangle$ with $x \in \mathcal{N}_{\varphi_{\Omega}}$. Hence the elements $(\iota \otimes \varphi_{\Omega})(\Delta_{\Omega}(y))$ and $\varphi_{\Omega}(y)1$ in the extended cone of M are equal on a normdense subset of M_*^+ . Since the latter element is bounded, the same is true of the former by lower-semi-continuity, and then their equality everywhere follows. □

Since the C*-algebra underlying a non-compact von Neumann algebraic quantum group is non-unital, we obtain the following corollary.

Corollary 2.12. *There exists a von Neumann algebraic quantum group (M, Δ) and a unitary 2-cocycle $\Omega \in M \otimes M$, such that the reduced (resp. universal) C^* -algebra associated to (M, Δ) is not isomorphic to the reduced (resp. universal) C^* -algebra associated to (M, Δ_Ω) , the Ω -twisted von Neumann algebraic quantum group.*

Remarks: 1. Note that the compact quantum group $\mathcal{L}^\infty(SU_q(2))$ is *not* a compact matrix quantum group. It would therefore be interesting to see if one can also twist compact *matrix* quantum groups into non-compact locally compact quantum groups. By the results of [5], this is closely related to the question whether a compact matrix quantum group can act ergodically on an infinite-dimensional type I-factor.

2. It follows from the results of [3] that if $(C(\mathfrak{G}), \Delta)$ is a compact quantum group, and Ω a unitary 2-cocycle *inside* $\text{Pol}(\mathfrak{G}) \odot \text{Pol}(\mathfrak{G})$, then the cocycle twisted von Neumann algebraic quantum group is again compact, with $\text{Pol}(\mathfrak{G})$ as the $*$ -algebra underlying the associated Hopf $*$ -algebra. Hence also the associated C^* -algebras remain unaltered.

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